Differential Groups and the Gamma Function

Michael F. Singer
(joint work with Charlotte Hardouin)
Mathematische Annalen, 342(2) 2008, 333-377

Department of Mathematics
North Carolina State University
Raleigh, NC 27695-8205
singer@math.ncsu.edu

KSDA, New York
12 December 2008
Theorem: (Hölder, 1887) The Gamma function $\Gamma(x + 1) = x\Gamma(x)$ satisfies no polynomial differential equation.

Theorem: (Hardouin, 2005; van der Put, 2006) Let $b(x) \in \mathbb{C}(x)$ and let $u(x)$ be a nonzero function, meromorphic on $\mathbb{C}$ such that

$$u(x + 1) = b(x)u(x).$$

The function $u(x)$ is differentially algebraic over 1-periodic meromorphic functions if and only if there exists a nonzero homogeneous linear differential polynomial $L(Y)$ with coefficients in $\mathbb{C}$ such that

$$L\left(\frac{b'(x)}{b(x)}\right) = g(x + 1) - g(x)$$

for some $g(x) \in \mathbb{C}(x)$.

Ex: For $\Gamma(x)$, $L\left(\frac{1}{x}\right) = g(x + 1) - g(x)$???

Also for $q$-difference equations and systems $u_i(x + 1) = b_i(x)u_i(x)$. 
Theorem: (Ishizaki, 1998) If \(a(x), b(x) \in \mathbb{C}(x)\) and \(z(x) \notin \mathbb{C}(x)\) satisfies
\[
z(qx) = a(x)z(x) + b(x), \ |q| \neq 1
\] (1)
and is meromorphic on \(\mathbb{C}\), then \(z(x)\) is not differentially algebraic over \(q\)-periodic functions.

Assume distinct zeroes and poles of \(a(x)\) are not \(q\)-multiples of each other.

Theorem: (H-S, 2007) \(z(x)\) is differentially algebraic iff \(a(x) = cx^n\) and
- \(b = f(qx) - a(x)f(x)\) for some \(f \in \mathbb{C}(x)\), when \(a \neq q^r\), or
- \(b = f(qx) - af(x) + dx^r\) for some \(f \in \mathbb{C}(x), d \in \mathbb{C}\) when \(a = q^r, r \in \mathbb{Z}\).
Theorem: (Roques, 2007) Let $y_1(x), y_2(x)$ be lin. indep. solutions of

$$y(q^2 x) - \frac{2a x - 2}{a^2 x - 1} y(q x) + \frac{x - 1}{a^2 x - q^2} y(x) = 0 \quad (2)$$

with $a \not\in q\mathbb{Z}$ and $a^2 \in q\mathbb{Z}$. Then $y_1(x), y_2(x), y_1(q x)$ are algebraically independent.

(H-S, 2007): $y_1(x), y_2(x), y_1(q x)$ are differentially independent. Give necessary and sufficient conditions for a large class of linear differential equations.
All of these results follow from a

Differential Galois Theory of Linear Difference Equations

and an understanding of

Linear Differential Groups
• Galois Theory of Linear Difference Equations

• Linear Differential Algebraic Groups

• Differential Galois Theory of Linear Difference Equations

• Differential Relations Among Solutions of Linear Difference Equations

• Final Comments
Galois Theory of Linear Difference Equations

\( k \) - field, \( \sigma \) - an automorphism  \( \text{Ex. } \mathbb{C}(x), \ \sigma(x) = x + 1, \ \sigma(x) = qx \)

Difference Equation: \( \sigma(Y) = AY \ \ A \in \text{GL}_n(k) \)

Splitting Ring: \( k\left[ Y, \frac{1}{\det(Y)} \right], \ \ Y = (y_{i,j}) \ \text{indeterminates} , \ \sigma(Y) = AY, \)

\( M = \max \sigma\text{-ideal} \)

\[ R = k\left[ Y, \frac{1}{\det(Y)} \right]/M = k\left[ Z, \frac{1}{\det(Z)} \right] = \sigma\text{-Picard-Vessiot Ring} \]

- \( M \) is radical \( \Rightarrow \) \( R \) is reduced

- If \( C = k^\sigma = \left\{ c \in k \mid \sigma c = c \right\} \) is alg closed \( \Rightarrow \) \( R \) is unique
  and \( R^\sigma = C \)

\( \text{Ex.} \)

\[ k = \mathbb{C} \ \ \sigma(y) = -y \]

\[ R = \mathbb{C}[y, \frac{1}{y}]/(y^2 - 1) \]
σ-Galois Group: \( \text{Gal}_{\sigma}(R/k) = \{ \phi : R \to R \mid \phi \text{ is a } \sigma \text{ } k\text{-automorphism} \} \)

Ex.

\[
k = \mathbb{C}, \quad \sigma(y) = -y \Rightarrow R = \mathbb{C}[y, \frac{1}{y}]/(y^2 - 1)
\]

\( \text{Gal}_{\sigma}(R/k) = \mathbb{Z}/2\mathbb{Z} \)

Ex.

\[
k = \mathbb{C}(x), \quad \sigma(x) = x + 1
\]

\( \sigma^2y - x\sigma y + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y \)

\( R = k[Y, \frac{1}{\det(Y)}]/(\det(Y) - 1), \quad \text{Gal}_{\sigma} = \text{SL}_2(\mathbb{C}) \)

Ex.

\[
\sigma(y) - y = f, \quad f \in k \Leftrightarrow \sigma \left( \begin{array}{cc} 1 & y \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} 1 & f \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & y \\ 0 & 1 \end{array} \right)
\]

\( \phi \in \text{Gal}_\sigma \Rightarrow \phi(y) = y + c_\phi, \quad c_\phi \in \mathbb{C} \)

\( \text{Gal}_\sigma = (\mathbb{C}, +) \text{ or } \{0\} \)
• $\phi \in \text{Gal}_\sigma, \ \sigma(Z) = AZ \Rightarrow \phi(Z) = Z[\phi], \ [\phi] \in \text{GL}_n(C)$

$\text{Gal}_\sigma \hookrightarrow \text{GL}_n(C)$ and the image is Zariski closed

$\text{Gal}_\sigma = G(C), \ G$ a lin. alg. gp. /C.

• $R =$ coord. ring of a $G$-torsor

$R^{\text{Gal}_\sigma} = k$

$\dim(G) = \text{Krull dim.}_k R \ (\simeq \text{trans. deg. of quotient field})$
The structure of $\text{Gal}(K/k)$ measures algebraic relations among the solutions.

Ex.

$$\sigma^2 y - x \sigma y + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y$$

$\text{Gal}_\sigma = \text{SL}_2(\mathbb{C})$

$3 = \dim \text{SL}_2(\mathbb{C}) = \text{tr. deg.}_k k(y_1, y_2, \sigma(y_1), \sigma(y_2))$

$\Rightarrow y_1, y_2, \sigma(y_1)$ alg. indep. over $k$
Ex. $f_1, \ldots, f_n \in k$, $k$ a difference field w. alg. closed const.

$$\sigma(y_1) - y_1 = f_1$$
$$\vdots$$
$$\sigma(y_n) - y_n = f_n$$

Picard-Vessiot ring $= k[y_1, \ldots, y_n]$

Prop. $y_1, \ldots, y_n$ alg. dep. over $k$ if and only if

$$\exists g \in k \text{ and a const coeff. linear form } L \text{ s.t. } L(y_1, \ldots, y_n) = g$$

(equiv., $c_1 f_1 + \ldots + c_n f_n = \sigma(g) - g$)

Ex. $y(x + 1) - y(x) = \frac{1}{x}$

$\frac{1}{x} \neq g(x + 1) - g(x) \Rightarrow y(x)$ is not alg. over $\mathbb{C}(x)$. 
Linear Differential Algebraic Groups

P. Cassidy—“Differential Algebraic Groups” Am. J. Math. 94(1972), 891-954 + 5 more papers, book by Kolchin, papers by Sit, Buium, Pillay et al., Ovchinnikov

\((k, \delta)\) = a differentially closed differential field

Definition: A subgroup \(G \subset GL_n(k) \subset k^{n^2}\) is a linear differential algebraic group if it is Kolchin-closed in \(GL_n(k)\), that is, \(G\) is the set of zeros in \(GL_n(k)\) of a collection of differential polynomials in \(n^2\) variables.

Ex. Any linear algebraic group defined over \(k\), that is, a subgroup of \(GL_n(k)\) defined by (algebraic) polynomials, e.g., \(GL_n(k), SL_n(k)\)

Ex. Let \(C = \ker \delta\) and let \(G(k)\) be a linear algebraic group defined over \(k\). Then \(G(C)\) is a linear differential algebraic group (just add \(\{\delta y_{i,j} = 0\}_{i,j=1}^n\) to the defining equations!)
Ex. Differential subgroups of $G_a(k) = (k, +) = \{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in k \}$

The linear differential subgroups are all of the form

$$G_a^L = \{ z \in k \mid L(z) = 0 \}$$

where $L$ is a linear homogeneous differential polynomial.

For example, if $m = 1$,

$$G_a^\delta = \{ z \in k \mid \delta(z) = 0 \} = G_a(C)$$

Ex. Differential subgroups of $G_a^n(k) = (k^n, +)$

The linear differential subgroups are all of the form

$$G_a^\mathcal{L} = \{(z_1, \ldots, z_n) \in k^n \mid L(z_1, \ldots, z_n) = 0 \ \forall L \in \mathcal{L} \}$$

where $\mathcal{L}$ is a set of linear homogeneous differential polynomials.
**Ex.** Differential subgroups of $G_m(k) = (k^*, \cdot) = GL_1(k)$

The connected linear differential subgroups are all of the form

$$G_a^L = \{ z \in k^* \mid L\left(\frac{z'}{z}\right) = 0 \}$$

where $L$ is a linear homogeneous differential polynomial.

This follows from the exactness of

$$(1) \longrightarrow G_m(C) \longrightarrow G_m(k) \overset{z \mapsto \frac{\partial z}{z}}{\longrightarrow} G_a(k) \longrightarrow (0)$$
Ex.  \( H \) a Zariski-dense proper differential subgroup of \( \text{SL}_n(k) \)
\[ \Rightarrow \exists g \in \text{SL}_n(k) \text{ such that } gHg^{-1} = \text{SL}_n(C), \ C = \ker(\delta). \]

In general if \( H \) a Zariski-dense proper differential subgroup of \( G \subset \text{GL}_n(k) \), a simple noncommutative algebraic group defined over \( C \)
\[ \Rightarrow \exists g \in \text{GL}_n(k) \text{ such that } gHg^{-1} = G(C), \ C = \ker(\delta). \]
Differential Galois Theory of Linear Difference Equations

$k$ - field, $\sigma$ - an automorphism  \( \delta \) - a derivation s.t. \( \sigma \delta = \delta \sigma \)

**Ex.** \( \mathbb{C}(x) : \sigma(x) = x + 1, \delta = \frac{d}{dx} \)

\[
\sigma(x) = qx, \quad \delta = x \frac{d}{dx}
\]

\( \mathbb{C}(x, t) : \sigma(x) = x + 1, \delta = \frac{\partial}{\partial t} \)

Difference Equation: \( \sigma(Y) = AY \quad A \in \text{GL}_n(k) \)

Splitting Ring: \( k \{ Y, \frac{1}{\det(Y)} \} = k[ Y, \delta Y, \delta^2 Y, \ldots, \frac{1}{\det(Y)} ] \)

\( Y = (y_{i,j}) \) differential indeterminates

\( \sigma(Y) = AY, \quad \sigma(\delta Y) = \delta(\sigma Y) = A(\delta Y) + (\delta A)Y, \ldots M = \max \sigma\delta\text{-ideal} \)

\[
R = k \{ Y, \frac{1}{\det(Y)} \} / M = k \{ Z, \frac{1}{\det(Z)} \} = \sigma\delta\text{-Picard-Vessiot Ring}
\]
$k$ - $\sigma\delta$ field

$\sigma(Y) = AY$, $A \in \text{GL}_n(k)$

$R = k\{Z, \frac{1}{\det(Z)}\}$ - $\sigma\delta$-Picard-Vessiot ring

- $R$ is reduced

- If $C = k^\sigma = \{c \in k \mid \sigma c = c\}$ is differentially closed
  $\Rightarrow R$ is unique and $R^\sigma = C$
$\sigma\delta$-Galois Group: $\text{Gal}_{\sigma\delta}(R/k) = \{\phi : R \to R \mid \phi \text{ is a } \sigma\delta \text{ } k\text{-automorphism}\}$

- $\phi \in \text{Gal}_{\sigma\delta}$, $\sigma(Z) = AZ \Rightarrow \phi(Z) = Z[\phi]$, $[\phi] \in \text{GL}_n(C)$
  $\text{Gal}_{\sigma\delta} \hookrightarrow \text{GL}_n(C)$ and the image is Kolchin closed
  $\text{Gal}_{\sigma\delta} = G(C)$, $G$ a lin. differential alg. gp. /$C$.

- $\text{Gal}_{\sigma\delta}$ is Zariski dense in $\text{Gal}_\sigma$

- $R =$ coord. ring of a $G$-torsor
  - $R^{\text{Gal}_{\sigma\delta}} = k$
  - Assume $G$ connected. Then diff. dim.$_C(G) =$ diff. tr. deg$_k F$
    where $F$ is the quotient field of $R$. 
\[k = \tilde{C} \quad \sigma(y) = -y \Rightarrow R = k[y, \frac{1}{y}]/(y^2 - 1)\]

\[\text{Gal}_{\sigma\delta}(R/k) = \mathbb{Z}/2\mathbb{Z}\]

\[\sigma(y) - y = f, \quad f \in k, \quad \text{Gal}_{\sigma \delta} \subset G_a\]

\[\Rightarrow \text{Gal}_{\sigma \delta} = \{c \in R^\sigma \mid L(c) = 0\} \text{ for some } L \in R^\sigma[\delta].\]

\[k = \tilde{C}(x), \quad \sigma(x) = x + 1, \quad \delta(x) = 1\]

\[\sigma^2 y - xy + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y\]

Will show: \( R = k\{Y, \frac{1}{\text{det}(Y)}\}/\{\text{det}(Y) - 1\}\)

\[\text{Gal}_{\delta \sigma} = \text{SL}_2(\tilde{C})\]
Differential Relations Among Solutions of Linear Difference Equations

Groups Measure Relations

\[ k - \sigma \delta - \text{field, } C = k^\sigma \text{ differentially closed.} \]

Differential subgroups of \( G_a^n(k) = (k^n, +) \) are all of the form

\[
G_a^L = \{(z_1, \ldots, z_n) \in k^n \mid L(z_1, \ldots, z_n) = 0 \ L \in \mathcal{L}\}
\]

where \( \mathcal{L} \) is a set of linear homogeneous differential polynomials.

\[
\Downarrow
\]

Proposition. Let \( R \) be a \( \sigma \delta \)-Picard-Vessiot extension of \( k \) containing \( z_1, \ldots, z_n \) such that

\[
\sigma(z_i) - z_i = f_i, \quad i = 1, \ldots, n.
\]

with \( f_i \in k \). Then \( z_1, \ldots, z_n \) are differentially dependent over \( k \) if and only if there is a homogeneous linear differential polynomial \( L \) over \( C \) such that

\[
L(z_1, \ldots, z_n) = g \quad g \in k
\]

Equivalently, \( L(f_1, \ldots, f_n) = \sigma(g) - g \).
Corollary. Let $f_1, \ldots, f_n \in \mathbb{C}(x), \sigma(x) = x + 1, \delta = \frac{d}{dx}$ and let $z_1, \ldots, z_n$ satisfy

$$\sigma(z_i) - z_i = f_i, \quad i = 1, \ldots, n.$$ 

Then $z_1, \ldots, z_n$ are differentially dependent over $\mathcal{F}(x)$ ($\mathcal{F}$ is the field of 1-periodic functions) if and only if there is a homogeneous linear differential polynomial $L$ over $\mathbb{C}$ such that

$$L(z_1, \ldots, z_n) = g \quad g \in \mathbb{C}(x)$$

Equivalently, $L(f_1, \ldots, f_n) = \sigma(g) - g$.

- Similar result for $q$-difference equations. Also for $\sigma y_i = f_i y_i$
The Gamma function is hypertranscendental.

• $z(x) = \Gamma'(x)/\Gamma(x)$ satisfies $\sigma(z) - z = \frac{1}{x}$.

• If $z(x)$ satisfies a polynomial differential equation, then

  $\exists L \in \mathbb{C}[\frac{d}{dx}], g(x) \in \mathbb{C}(x)$ s.t. $L\left(\frac{1}{x}\right) = g(x + 1) - g(x)$

• $L\left(\frac{1}{x}\right)$ has a pole $\Rightarrow g(x)$ has a pole.

• If $g(x)$ has a pole then $g(x + 1) - g(x)$ has at least two poles but $L\left(\frac{1}{x}\right)$ has exactly one pole.
If $H$ a Zariski-dense proper differential subgroup of $G \subset \text{GL}_n(k)$, a simple noncommutative algebraic group defined over $C$

$$\Rightarrow \exists g \in \text{GL}_n(k) \text{ such that } gHg^{-1} = G(C), \ C = \ker(\delta).$$

Proposition. Let $A \in \text{GL}_n(k)$ and assume the $\sigma$-Galois group of $\sigma(Y) = AY$ to be a simple noncommutative linear algebraic group $G$ of dimension $t$. Let $R = k\{Z, \frac{1}{\det Z}\}$ be the $\sigma\delta$-PV ring.

The differential trans. deg. of $R$ over $k$ is less than $t$

$$\Leftrightarrow$$

$$\exists B \in \text{gl}_n(k) \text{ s.t. } \sigma(B) = ABA^{-1} + \delta(A)A^{-1}$$

(in which case, $(\delta Z - BZ)Z^{-1} \in \text{gl}_n(k^\sigma)$)
Ex.

\[ k = \mathbb{C}(x), \sigma(x) = x + 1 \]

\[ \sigma^2 y - x \sigma y + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y \]

\[ R = k[Y, \frac{1}{\det(Y)}]/(\det(Y) - 1), \quad \text{Gal}_\sigma = \text{SL}_2(\mathbb{C}) \]

\[ y_1(x), y_2(x) \text{ linearly independent solutions.} \]

\[ y_1(x), y_2(x), y_1(x + 1) \text{ are differentiably dependent over } \mathbb{C}(x) \]

\[ \iff \exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{gl}_2(\mathbb{C}(x)) \text{ s.t.} \]

\[ \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left( \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}' \right) \left( \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}^{-1} \right) + \left( \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} \right) \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \left( \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}^{-1} \right) \]

This 4\textsuperscript{th} order inhomogeneous equation has no such solutions

\[ \Rightarrow y_1(x), y_2(x), y_1(x + 1) \text{ are differentiably independent over } \mathbb{C}(x) \]
Final Comments

• General Theory: Consider integrable

$$\Sigma = \{\sigma_1, \ldots, \sigma_r\}, \Delta = \{\partial_1, \ldots, \partial_s\}$$

linear systems and measure dependence on auxillary derivations \(\delta_1, \ldots, \delta_t\). Can show that for

$$\gamma(x, t) = \int_1^t u^x e^{-u} du$$

we have \(\gamma, \gamma_x, \gamma_{xx}, \ldots\) alg. indep \(\mathbb{C}(x, t)\).

• Isomonodromic \(\Leftrightarrow\) constant Galois group.

• Inverse problem

• Nonlinear equations