Notes on Malgrange’s General Involutivity Theorem [1]

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The setting is a fiber bundle $\pi : E \to M$ of manifolds modeled over $\mathbb{C}$ ($\dim M = n$, $\dim E = n + m$). Let $Z \subseteq J^k\pi$ be a locally closed embedded sub-manifold (a $k$-th order differential equation); and, fix $p \in Z$ together with a adapted chart $(U, (x^i, u^\alpha, u_i^\alpha))$ around $p$ of $J^k\pi$. We denote $x^i(p) = p^i$, $u^\alpha(p) = q^\alpha$, $u_i^\alpha(p) = q_i^\alpha$.

Let $I$ be the sheaf of ideals defined over $U$ of sections of $\mathcal{O}_{J^k\pi}|_U$ vanishing on $U \cap Z$. Note that if $F \in I_p$ does not involve the $u_i^\alpha$, $|I| = k$, then $D_i F \in I_p$.

1 Setting the necessary and sufficient conditions for a lifting of $p$ to exist.

For $F \in \mathcal{I}_p$, set

$$D_i F(p) = \frac{\partial F}{\partial x_i}(p) + \sum_{\alpha, |I| < k} \frac{\partial F}{\partial u_i^\alpha}(p) q_i^\alpha + \sum_{\alpha, |I| = k} \frac{\partial F}{\partial u_i^\alpha}(p) u_i^\alpha,$$

where $D_i F(p)$ is a complex number and $D_i F(p)(u_i^\alpha)$ is a linear form on the complex vector space with coordinates $\{u_i^\alpha\}$. The point $p \in Z$ (the $k$-th order jet solution to the differential equation) can be lifted to the first prolongation if and only if the system of equations

$$D_i F(p)(u_i^\alpha) = -D_i F(p), \quad F \in \mathcal{I}_p, \quad i \in \{1, \ldots, n\}$$

has a solution. Alternatively, if we take a set of generators of $\mathcal{I}_p$, $F_1, \ldots, F_r$, then to solve the previous system of equation is equivalent the following system

$$D_i F_j(p)(u_i^\alpha) = -D_i F_j(p), \quad j \in \{1, \ldots, r\}, \quad i \in \{1, \ldots, n\}.$$

When one allows $i$ to swipe the whole interval $\{1, \ldots, n\}$ the association:

$$(u_i^\alpha)_{i, \alpha, |I| = k} \mapsto (D_i F(p)(u_i^\alpha))_i$$
is a map from the complex vector space with coordinates \( \{ u^\alpha_{I+\epsilon_i} \}_{i,\alpha,|I|=k} \) into \( \mathbb{C}^n \). So the system of equation has a solution if and only if each vectors \((D'_i F_j(p))_{i}\) is a zero of the linear forms annihilating the image of the map \((u^\alpha_{I+\epsilon_i}) \mapsto (D''_i F(p)(u^\alpha_{I+\epsilon_i})))

Namely, the system has a solution if and only if:

\[
\sum_{i,j} \lambda^{i,j} D''_i F_j(p) = 0, \quad \lambda^{i,j} \in \mathbb{C} \implies \sum_{i,j} \lambda^{i,j} D'_i F_j(p) = 0
\]

2 Writing the conditions in terms of Koszul complexes.

Denote by \( A = \mathbb{C}[\xi_1, \ldots, \xi_n] \) the ring of polynomial in \( n \)-variables over \( \mathbb{C} \). We look at \( A \) as a ring graded by degree, and we put \( T = A_0 = \mathbb{C} \xi_1 + \ldots + \mathbb{C} \xi_n \).

We make from the free module of rank \( m \) over \( \mathcal{O}_{J^k \pi, p} \) an \( A \)-module, \( B \), by tensoring it. Explicitly:

\[
B = \bigoplus_{\alpha} \mathcal{O}_{J^k \pi, p} \delta u^\alpha \otimes \mathbb{C} A \\
\simeq \bigoplus_{\alpha, |I|=k} \mathcal{O}_{J^k \pi, p} \xi_1, \ldots, \xi_n \delta u^\alpha
\]

where \( \delta u^1, \ldots, \delta u^m \) is a free basis. In particular, conveying the notation

\[
\xi_I \delta u^\alpha =: \delta u^\alpha_I,
\]

where \( I \) is a multi-index, \( B \) has a natural graded \( A \)-module structure

\[
B = \bigoplus_{l \geq 0} \left( \bigoplus_{\alpha, |I|=l} \mathcal{O}_{J^k \pi, p} \delta u^\alpha_I \right)
\]

Given \( F \in \mathcal{O}_{J^k \pi, p} \) we define the \((k\text{-th order})\) symbol of \( F \) to be the element

\[
\delta F = \sum_{\alpha, |I|=k} \frac{\partial F}{\partial u^\alpha_I} \delta u^\alpha_I
\]

(“the differential of \( F \) modulo the \( dx^i \) and the \( du^\alpha_I, |I| < k \).”)

Going back to the \( D''_i F(p) \), we express the identity \( \sum_{i,j} \lambda^{i,j} D''_i F_j(p) = 0 \) in
terms of the Koszul complex:

\[
0 = \sum_{i,j} \lambda_{i,j} D_i^r F_j(p)
\]

\[
0 = \sum_{i,j} \lambda_{i,j} \left( \sum_{\alpha,|I|=k} \frac{\partial F_j}{\partial u_I^\alpha}(p) \delta u_{I+\epsilon_i}^{\alpha} \right)
\]

\[
= \sum_{i,j} \lambda_{i,j} \left( \xi_i \sum_{\alpha,|I|=k} \frac{\partial F_j}{\partial u_I^\alpha}(p) \delta u_I^{\alpha} \right)
\]

\[
= \sum_{i} \xi_i \sum_{j} \lambda_{i,j} \delta F_j(p)
\]

\[
= d(\sum_{i} \xi_i \otimes \sum_{j} \lambda_{i,j} \delta F_j(p))
\]

where \(d\) is the boundary operator in the Koszul complex \(K_*(\xi, B)\):

\[
d : \left( \bigwedge^p T \right) \otimes B \rightarrow \left( \bigwedge^{p-1} T \right) \otimes B
\]

\[
\xi_i \wedge \ldots \wedge \xi_i \otimes g \rightarrow \sum_{j=1}^{p} (-1)^{j+1} \xi_i \wedge \ldots \wedge \xi_j \wedge \ldots \wedge \xi_i \otimes \xi_i g
\]

So that

\[
\sum_{i,j} \lambda_{i,j} D_i^r F_j(p) = 0 \iff \sum_{i} \xi_i \otimes \left( \sum_{j} \lambda_{i,j} \delta F_j \right)(p) \in \mathbb{Z}[K_1(\xi, B_k)(p)]
\]

On the other hand the identity \(\sum_{i,j} \lambda_{i,j} D_i^r F_j(p) = 0\) is equivalent to the fact that in the expression \(\sum_{i,j} \lambda_{i,j} D_i^r F_j(p)\) the \(u_{I+\epsilon_i}^{\alpha}\) vanishes, in particular

\[
\sum_{i,j} \lambda_{i,j} D_i^r F_j(p) = \sum_{i,j} \lambda_{i,j} D_i^r F_j(p).
\]

Now we consider the situation over \(Z\) around \(p\). So instead of \(B\) we consider

\[
B/\mathcal{J}_p = \bigoplus_{l \geq 0} (\oplus_{\alpha,|I|=l} \mathcal{O}_{Z,\delta u_I^{\alpha}})
\]

and we set \(N\) to be the sub-module of \(B/\mathcal{J}_p\) generated by the \((k\)-th order\) symbols \((\text{mod } \mathcal{J}_p)\) of the \(F \in \mathcal{J}_p\). And we define the torsion of \(Z\) at \(p\) to be the map:

\[
\tau_p : \mathbb{Z}[K_1(\xi, N_k)] \rightarrow \mathbb{C}
\]

\[
\sum_{i} \xi_i \otimes \delta g_i \rightarrow \sum_{i} D_i g_i(p)
\]

So the discussion above means \(p\) can be lifted if and only if \(\tau_p \equiv 0\).
3 Two remarks

3.1

Let \( N' \) be the sub-module of generated \( B/I_p \) generated by the \( k-1 \)st order
symbols (mod \( I_p \)) of the \( F \in \mathcal{I}_p \) not involving the \( u^I_\alpha \), \(|I| = k \). Then:

**Fact:** \( \tau_p \) vanishes in the image of \( d : \wedge^2 T \otimes N_{k-1}' \to T \otimes N_k \).

Indeed

\[
\tau_p(d(\xi_i \wedge \xi_j \otimes \delta g)) = \tau_p(\xi_j \otimes \delta D_i g - \xi_i \otimes \delta D_j g) = (D_j D_i g - D_i D_j g)(p) = 0
\]

3.2

**Fact:** \( H_p(K_\bullet(\xi, N_l)(p)) = \bigoplus_l H_p(K_\bullet(\xi, N_l)(p)) \) is a finite dimensional complex vector space.

So that \( H_1(K_\bullet(\xi, N_l)(p)) = 0 \) for \( l >> 0 \); or equivalently (snake lemma), \( H_2(K_\bullet(\xi, M_l)(p)) = 0 \) for \( l >> 0 \), where \( M \) is defined by:

\[
0 \to N \to B/I_p \to M/ \to 0
\]

References