On Partitioned Differential Quasifields

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We begin by recalling that a differential ring $R$ is called a (differential) quasi-field if every nonunit in $R$ is nilpotent and every nonzero element has some derivative (perhaps of order zero) that is not nilpotent. Every differential field is a differential quasifield, and in characteristic zero, every differential quasifield is a differential field. Also, the subring of constants of a differential quasifield is a field.

We also recall that there are equivalent ways of describing a differential quasifield, which illustrate the parallel between fields and differential quasifields.
Recall that a ring $R$ is **reduced** if $R$ has no nonzero nilpotents, and $R$ is **pointed** if every nonunit in $R$ is nilpotent. Observe that a ring $R$ is a field if and only if $R$ is reduced and pointed. An element $x$ in a differential ring $R$ is called **differentially nilpotent** if all order derivatives of $x$ (including order zero) are nilpotent in $R$. Note that any differentially nilpotent element is nilpotent. A differential ring $R$ is called **quasireduced** if $R$ has no nonzero differentially nilpotent elements. Hence a
differential ring $R$ is a differential quasi-field if and only if $R$ is quasireduced and pointed.

**Example 1** Consider the differential ring $A = \mathbb{Z}\{y\}$ of differential polynomials with integer coefficients in the differential indeterminate $y$. In $A$, consider the differential ideal $Q$ generated by $\{y' - 1, y^2, 2\}$, and let $R = A/Q$. If we let $x$ denote the element $y + Q \in R$, then we see that $R = \{0, 1, x, 1 + x\}$ has characteristic 2, that $x^2 = 0$ and $x' = 1$. Hence $R$ is quasireduced and
pointed, i.e., $R$ is a differential quasifield.

**Proposition 2** Suppose that $E$ and $F$ are differential quasifields, and that $f : E \rightarrow F$ is a differential ring homomorphism. Then if $C_E$ denotes the subfield of constants in $E$, we have:

1. $f$ is injective.

2. $x \in E$ is nilpotent if and only if $f(x) \in F$ is nilpotent.
3. $x \in E$ is constant if and only if $f(x) \in F$ is constant.

4. $x \in E$ is invertible if and only if $f(x) \in F$ is invertible.

5. $\{x_1, \ldots, x_n\} \subseteq E$ is linearly independent over $C_E$ if and only if
$\{f(x_1), \ldots, f(x_n)\} \subseteq F$ is linearly independent over $f(C_E)$.

**Theorem 3** Let $E$ be a differential quasi-field of characteristic $p > 0$ with derivation $\delta$ and field of constants $C_E$. If
φ : E → E is any differential automorphism that leaves $C_E$ fixed, then
φ = id_E, the identity on E.

Proof: Let $x \in E$, then $x^p \in C_E$ because $\delta(x^p) = px^{p-1}\delta(x) = 0$. If $\bar{x} = \phi(x) - x$, then $\bar{x}$ is nilpotent, since

\[
\bar{x}^p = (\phi(x) - x)^p = \phi(x)^p - x^p = \phi(x^p) - x^p = x^p - x^p = 0.
\]
Now for any $m \in \mathbb{N}$, we have

$$(\delta^m(x))^p = (\delta^m(\phi(x) - x))^p$$

$$= (\delta^m(\phi(x)) - \delta^m(x))^p$$

$$= (\phi(\delta^m(x)) - \delta^m(x))^p$$

$$= (\phi(\delta^m(x))^p - (\delta^m(x))^p$$

$$= \phi((\delta^m(x))^p) - (\delta^m(x))^p$$

$$= (\delta^m(x))^p - (\delta^m(x))^p$$

$$= 0.$$ 

Thus for all $m \in \mathbb{N}$, $\delta^m(x)$ is nilpotent. But $E$ is a quasifield, so it must be that $\bar{x} = 0$, that is, $\phi(x) = x$. $\square$
We recall that for any commutative ring $R$ with identity, the \textbf{ring of Hurwitz series} over $R$, denoted by $HR$, is defined as follows. The elements of $HR$ are sequences $(a_n) = (a_0, a_1, a_2, \ldots)$, where $a_n \in R$ for each $n \in \mathbb{N}$. Let $(a_n), (b_n) \in HR$. Addition in $HR$ is defined termwise, i.e.,

$$(a_n) + (b_n) = (c_n), \quad \text{where} \quad c_n = a_n + b_n$$

for all $n \in \mathbb{N}$. The (Hurwitz) product of $(a_n)$ and $(b_n)$ is given by $(a_n) \cdot (b_n) = (c_n)$, where

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$
Moreover, $HR$ is a differential ring with derivation $\partial_R : HR \to HR$ given by

$$\partial_R((a_0, a_1, a_2, \ldots)) = (a_1, a_2, a_3, \ldots).$$

We will often write $\partial$ in place of $\partial_R$. We will denote, for any $j \in \mathbb{N}$, the additive mapping $\pi_j : HR \to R$ defined by $\pi_j((a_n)) = a_j$.

Recall also that for any ring $R$ of positive characteristic, $R$ is a field if and only if the differential ring $HR$ of Hurwitz series over $R$ is a differential quasi-field.
Proposition 4 Let $E$ be a differential quasifield of positive characteristic, $N_E$ the nilradical of $E$ and $C_E$ the subfield of constants in $E$. Then there is a natural injective differential ring homomorphism $\eta_E : E \to Hk$ of $E$ into the quasifield of Hurwitz series $Hk$, where $k = E/N_E$. Moreover, we have:

1. $x \in E$ is invertible in $E$ if and only if $\eta_E(x) \in Hk$ is invertible in $Hk$.

2. $x \in E$ is nilpotent in $E$ if and only if $\eta_E(x) \in Hk$ is nilpotent in $Hk$. 
3. $x \in E$ is constant in $E$ if and only if
   \[ \eta_E(x) \in Hk \text{ is constant in } Hk. \]

4. Let $X = \{x_1, x_2, \ldots, x_n\} \subseteq E$. Then
   $X$ is linearly independent over $C_E$ if and only if
   \[ \eta_E(X) \subseteq Hk \text{ is linearly independent over } \eta_E(C_E). \]

Here $\eta_E$ is defined by
\[
\eta_E(x) = (x + N_E, \delta_E(x) + N_E, \delta_E^2(x) + N_E, \ldots).\]
Let $E$ be a differential quasifield, $C_E$ the subfield of constants of $E$, and $N_E$ the nilradical of $E$. We say that $E$ is **partitioned** if, as an additive group, $E = C_E \oplus N_E$, i.e., if for any $x \in E$, there exist unique $c_x \in C_E$ and $n_x \in N_E$ such that $x = c_x + n_x$.

**Proposition 5** Let $E$ be a differential quasifield with positive characteristic $p$ and field of constants $C_E$. If $C_E$ is perfect, then $E$ is partitioned.

**Proof:** First recall that if $x \in E$, then $x^p \in C_E$. Now let $N_E$ be the nilradical
of \( E \) and let

\[
k = E/N_E.
\]

Then \( k \) is an extension of \( C_E \), in the obvious way, that is,

\[
0 \rightarrow C_E \rightarrow E \rightarrow k.
\]

If

\[
x \in k \quad \text{then} \quad x^p \in C_E,
\]

and thus, since \( C_E \) is perfect,

\[
k = C_E.
\]

Thus \( N_E \) has codimension one, in \( E \), as a \( C_E \)-vector space and so

\[
E = C_E \oplus N_E,
\]
that is, $E$ is partitioned.

\[\square\]

**Proposition 6** Let $(E, \delta_E)$ be a differential quasifield of positive characteristic, $C_E$ the subfield of constants of $E$, $N_E$ the nilradical of $E$, 

\[k = E/N_E\]

the reduced field of $E$ and 

\[\eta : E \rightarrow H_k\]

the canonical embedding. Then $E$ is partitioned if and only if 

\[\eta(C_E) = C_{H_k} \cong k.\]
**Proof:** Suppose first that $E$ is partitioned, and let

$$c = (c_0, 0, \ldots, 0, \ldots) \in C_{Hk},$$

so that $c_0 \in k = E/N_E$. Hence there exists $x \in E$ such that

$$\tau(x) = c_0,$$

where $\tau : E \to k$ is the canonical surjection. Since $E$ is partitioned, there exists $c_x \in C_E$ such that $x - c_x \in N_E$. Then

$$\eta(c_x) = (\tau(c_x), \tau(\delta_E(c_x)), \tau(\delta_E^2(c_x)), \ldots) = (c_0, 0, \ldots, 0, \ldots),$$
showing that $\eta(C_E) = C_{Hk} \cong k$.

Conversely, suppose that $\eta(C_E) = C_{Hk}$, so that for any $x \in E$, there exists $c_x \in C_E$ such that

$$\eta(c_x) = (x + N_E, 0 + N_E, 0 + N_E, \ldots) \in C_{Hk}.$$ 

It is clear that $c_x \in C_E$ satisfying $\eta(c_x) = (x + N_E, 0 + N_E, 0 + N_E, \ldots)$ is unique, since $\eta$ is injective. Also, $x - c_x$ is nilpotent in $E$, since
\[ \eta(x - cx) = (x + NE, \delta_E(x) + NE,\]
\[ \delta_E^2(x) + NE, \ldots) \]
\[ - (x + NE, 0 + NE,\]
\[ 0 + NE, \ldots) \]
\[ = (0 + NE, \delta_E(x) + NE,\]
\[ \delta_E^2(x) + NE, \ldots) \]

is nilpotent in \( Hk \). Hence \( E \) is partitioned. \qed

A basic result in differential algebra (of characteristic zero) is that if \((F, \delta_F)\) is a differential field of characteristic
zero with field of constants $C$, then $y_1, \ldots, y_n \in F$ are linearly dependent over $C$ if and only if the Wronskian $w(y_1, \ldots, y_n) = 0$. The Wronskian $w(y_1, \ldots, y_n)$ of $y_1, \ldots, y_n \in F$ is defined by

$$w(y_1, \ldots, y_n) = \det(\delta_F^{-1}(y_j)).$$

This result does not carry over directly to differential quasifields of positive characteristic, as the following example shows.

**Example 7** Let $k$ be a field of characteristic 2, and consider the differential quasifield $Hk$. The elements $x^{[2]}$
and $x^{[3]}$ in $Hk$ are certainly linearly independent over $C_{Hk} \cong k$, but a quick calculation using

$$x^{[m]}x^{[n]} = \binom{m+n}{n}x^{[m+n]}$$

shows that

$$w(x^{[2]}, x^{[3]}) = 0.$$ 

In order to generalize this result about linear dependence over constants to the case of differential quasifields, we need to introduce the following.
Let \((R, \delta_R)\) be any differential ring, let 
\[ y = (y_1, \ldots, y_n) \in R^n, \]
and let 
\[ s = (s_1, \ldots, s_n) \in \mathbb{N}^n. \]

The \(s\) – quasiwronskian of \(y\), denoted by \(w_s(y)\), is defined by 
\[
    w_s(y) = \det(\delta^s_R(y_j)).
\]
So the (usual) Wronskian of \(y\) is the \((0, 1, \ldots, n - 1)\)-quasiwronskian of \(y\).

**Theorem 8** Let \(E\) be a differential quasi-field of positive characteristic with field of constants \(C_E\), suppose that \(E\) is partitioned, and let 
\[ y = (y_1, \ldots, y_n) \in E^n. \]
Then \( \{y_1, \ldots, y_n\} \) is linearly independent over \( C_E \) if and only if there exists \( s = (s_1, \ldots, s_n) \in \mathbb{N}^n \) such that \( w_s(y) \) is invertible in \( E \).

**Proof:** Assume first that \( \{y_1, \ldots, y_n\} \) is linearly dependent over \( C_E \), so that there exist \( c_1, \ldots, c_n \in C_E \), not all zero, such that \( \sum_{j=1}^{n} c_j y_j = 0 \). Hence for any \( s = (s_1, \ldots, s_n) \in \mathbb{N}^n \), \( (c_1, \ldots, c_n) \) is a non-trivial solution in \( C_E^n \) to the system of linear equations

\[
\sum_{j=1}^{n} \partial_s^i(y_j)x_j = 0, \quad i = 1, \ldots, n,
\]
with coefficients in $E$ in the unknowns $x_1, \ldots, x_n$. The determinant of the matrix of coefficients of the above system is the $s$-quasiwronskian $w_s(y)$ of $y = (y_1, \ldots, y_n)$, and since this system has a non-trivial solution, this determinant is not invertible in $E$, and hence is nilpotent in $E$.

Now assume that $\{y_1, \ldots, y_n\}$ is linearly independent over $C_E$. We proceed with a special case, namely when $E = Hk$ for a field $k$ of positive characteristic.
Lemma 9 Let $k$ be a field of positive characteristic, $E = Hk$ the differential quasifield of Hurwitz series over $k$, and $(y_1, \ldots, y_n) \in E^n$. If $\{y_1, \ldots, y_n\} \subseteq E$ is linearly independent over $k$, then there exists some $(s_1, \ldots, s_n) \in \mathbb{N}^n$ such that $\det(\partial^{s_i}(y_j))$ is invertible in $E$.

Proof: We proceed using induction on $n$. If $n = 1$, then $y_1$ is linearly independent over $k$ if and only if $y_1 \neq 0$, so take $s_1 = \text{ord}(y_1)$. Then $\partial^{s_1}(y_1) = \det(\partial^{s_1}(y_1))$ is invertible in $E$, since

$$\pi_{s_1}(y_1) = \pi_0(\partial^{s_1}(y_1)) \neq 0.$$
Now suppose that \((y_1, \ldots, y_n) \subseteq E\) is linearly independent over \(k\). We may also assume that \(s_1 = \text{ord}(y_1) \leq \text{ord}(y_j)\) for \(j = 2, \ldots, n\). Define

\[ c_j = \pi_{s_1}(y_j)\pi_{s_1}(y_1)^{-1} \in k \]

for \(j = 2, \ldots, n\), and define

\[ z_1 = y_1 \quad \text{and} \quad z_j = y_j - c_j y_1 \]

for \(j = 2, \ldots, n\). A routine calculation shows that \(\{z_1, \ldots, z_n\}\) is linearly independent over \(k\). Furthermore, we see that \(\text{ord}(z_j) > s_1\) for \(j = 2, \ldots, n\), since
\[
\pi s_1(z_j) = \pi s_1(y_j - c_j y_1) \\
= \pi s_1(y_j) \\
- \pi s_1(y_j) \pi s_1(y_1)^{-1} \pi s_1(y_1) \\
= 0
\]

Since \( \{z_2, \ldots, z_n\} \) is linearly independent over \( k \), by induction there exists

\[
(s_2, \ldots, s_n) \in \mathbb{N}^{n-1}
\]

such that

\[
\text{det}((\partial^{s_i} y_j)_{2 \leq i,j \leq n})
\]
is invertible in $Hk$. Then since
\[ \det(\partial^s_i(y_j)) = \det(\partial^s_i(z_j)), \]
and since
\[ \partial^s_1(z_1) = \partial^s_1(y_1) \]
is invertible in $Hk$, we see by expanding
\[ \det(\partial^s_i(z_j)) \]
along the first row that
\[
\det(\partial^s_i(z_j)) = \partial^s_1(z_1) \det((\partial^s_i(z_j))_{2 \leq i, j \leq n}) \\
+ \sum_{j=2}^{n} (-1)^{j+1} \partial^s_1(z_j) M_{(1,j)},
\]
where $M_{(1,j)}$ is the $(1, j)$-minor of $(\partial^s_i(z_j))$. Now since each $\partial^s_1(z_j)$ is nilpotent in
For $j = 2, \ldots, n$, we see that
\[
\sum_{j=2}^{n} (-1)^{j+1} \partial^{s_1}(z_j) M_{(1,j)}
\]
is nilpotent in $Hk$, so that
\[
\det(\partial^{s_i}(z_j)) = \det(\partial^{s_i}(y_j))
\]
is invertible in $Hk$. □

Continuing with the proof of Theorem 8, let $k = E/N_E$, where $N_E$ is the nil-
radical of $E$, and consider the embedding $\eta_E : E \longrightarrow Hk$. By Proposition 4,
\( \{y_1, \ldots, y_n\} \subseteq E \) is linearly independent over \( C_E \) if and only if

\[ \{\eta(y_1), \ldots, \eta(y_n)\} \subseteq Hk \]

is linearly independent over \( \eta(C_E) \). Since \( E \) is partitioned

\[ \eta(C_E) \cong k \]

by Proposition 6, so Lemma 9 applies to show that there exists some

\[ (s_1, \ldots, s_n) \in \mathbb{N}^n \]

such that \( \det(\partial^{s_i}(\eta(y_j))) \) is invertible in \( Hk \). But since

\[ \det(\partial^{s_i}(\eta(y_j))) = \eta(\det(\partial^{s_i}(y_j))), \]
Proposition 2 tells us that \( \det(\partial^s_i(y_j)) \) is invertible in \( E \), as desired. \( \square \)

**Corollary 10** Let \( k \) be a field of positive characteristic, and let \( K \) be any field extension of \( k \). The finite set

\[
\{h_1, \ldots, h_n\} \subseteq Hk
\]

is linearly independent over \( k \) if and only if

\[
\{h_1, \ldots, h_n\} \subseteq HK
\]

is linearly independent over \( K \).
Proof: Clearly if \( \{h_1, \ldots, h_n\} \) is linearly dependent over \( k \), then \( \{h_1, \ldots, h_n\} \) is linearly dependent over \( K \). Now assume that \( \{h_1, \ldots, h_n\} \) is linearly independent over \( k \). By Theorem 8, there is some \( s \in \mathbb{N}^n \) such that \( w_s(h_1, \ldots, h_n) \) is invertible in \( k \), and hence is invertible in \( K \). It follows from Theorem 8 again that \( \{h_1, \ldots, h_n\} \) is linearly independent over \( K \). 

Recall that if \( A, B, \) and \( C \) are rings and if \( f : A \to C \) and \( g : B \to C \) are ring morphisms, then the bilinear mapping
\[ A \times B \longrightarrow C \] given by \((a, b) \mapsto f(a)g(b)\) induces a ring morphism

\[ \Phi : A \otimes B \longrightarrow C. \]

In addition if \(A, B\) and \(C\) are differential rings with derivations \(\delta_A, \delta_B\) and \(\delta_C\) respectively and \(f : A \longrightarrow C\) and \(g : B \longrightarrow C\) are differential ring morphisms then \(A \otimes B\) is a differential ring with derivation

\[ \delta_{A \otimes B} = \delta_A \otimes \text{id}_B + \text{id}_A \otimes \delta_B \]

and \(\Phi : A \otimes B \longrightarrow C\) is a differential ring morphism.
Proposition 11 Let $K$ be any field extension of $k$. Then the differential $k$-algebra homomorphism

$$\Phi : K \otimes_k Hk \rightarrow HK,$$

defined by

$$\Phi(a \otimes (b_n)) = (ab_n)$$

for any $a \in K$ and $(b_n) \in Hk$, is injective.

Proof: Here we must show that if

$$\sum_{i=1}^{n} a_i \otimes h_i \neq 0$$
with $a_i \in K$ and $h_i \in Hk$ then

$$
\sum_{i=1}^{n} a_i h_i \neq 0.
$$

Reduce to the case where $\{h_1, \ldots, h_n\}$ is linearly independent over $k$. Now, if we were to have

$$
\sum_{i=1}^{n} a_i h_i = 0,
$$

then since $\{h_1, \ldots, h_n\}$ is linearly independent over $k$, by Corollary 10 $\{h_1, \ldots, h_n\}$ is also linearly independent over $K$. Thus we must have

$$
a_i = 0
$$
for each $i$, but this implies that

$$\sum_{i=1}^{n} a_i \otimes h_i = 0.$$ 

\[ \square \]

**Theorem 12** Let $E$ be a partitioned differential quasifield and let $A$ be a differential ring that is obtained from $E$ by extension of scalars. Then $A$ is a partitioned differential quasifield.

**Proof:** Let $E$ have field of constants $C_E$, nilradical $N_E$ and let $C_E \subset K$ be a field extension. We have

$$A = K \otimes_{C_E} E$$
and since $E = C_E \oplus N_E$, it follows that

$$A = (K \otimes_{C_E} C_E) \oplus (K \otimes_{C_E} N_E)$$

as an abelian group. Now

$$K \cong K \otimes_{C_E} C_E,$$

so identify $K \otimes_{C_E} C_E$ with $K$. Set

$$N = K \otimes_{C_E} N_E.$$

Clearly $N$ is the nilradical of $A$ and

$$A = K \oplus N.$$

If $x \in A$ and $x \notin N$ then $x = r + n$ for $0 \neq r \in K$ and $n$ a nilpotent. If $n^m = 0$, 
then letting
\[ y = \frac{1}{r} \left( 1 + \sum_{i=1}^{m-1} \left( \frac{-n}{r} \right)^i \right) \]

it is easy to verify that \( xy = 1 \). Thus \( A \) is a differential ring such that each element is nilpotent or is invertible. We must show that for every nilpotent \( n \in A \), there is some \( l \in \mathbb{N} \) such that \( \delta^l_A(n) \) is invertible in \( A \), where \( \delta_A \) is the derivation on \( A \). Since \( E \) is partitioned,

\[ E/N_E \cong C_E, \]

and so we have an embedding

\[ \eta : E \rightarrow HC_E. \]
Since $K$ is a $C_E$-vector space, $K$ is flat as a $C_E$-module, and so

$$id_K \otimes \eta : A \longrightarrow K \otimes_{C_E} HC_E$$

is an embedding. Also

$$\Phi : K \otimes_{C_E} HC_E \longrightarrow HK$$

is an embedding by Proposition 11, so the composition

$$\vartheta = \Phi \circ id_K \otimes \eta, \quad \vartheta : A \longrightarrow HK$$

is an embedding. If $n \in A$ is a nonzero nilpotent, then $\vartheta(n)$ is a nonzero nilpotent in $HK$. Thus there is some $l \in \mathbb{N}$
such that $\partial^l_K \vartheta(n)$ is invertible, so from what we have already shown $\delta^l_A(n)$ is invertible. We conclude that $A$ is a quasifield. Since $A = K \oplus N$, $A$ is partitioned. \qed