Math A4400: Linear Algebra

7th homework set, due at 2pm on Tuesday, November 4th.

Bring your solutions class, or slide them under the door of my office NAC 6278.

Solve any 6 problems.

1. Using the formula for the inverse of a $2 \times 2$ matrix, find all eigenvalues and eigenvectors of the following matrices. For each of them, find a basis of $\mathbb{R}^2$ containing as many eigenvectors as possible, the change of basis matrix, and the similar matrix representing the same linear transformation with respect to this new basis.

   $A := \begin{bmatrix} 13 & -3 \\ 18 & -2 \end{bmatrix}$  \hspace{1cm} $B := \begin{bmatrix} 9.5 & -4.5 \\ 12.5 & -5.5 \end{bmatrix}$

2. Find a matrix $B \in \mathbb{C}^{4 \times 4}$ such that $B^2 = \begin{bmatrix} -9 & 0 & 39 & 13 \\ 0 & -9 & -13 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. Hint: problem set 6.

3. Let $V$ be the real vector space of infinitely-differentiable functions $f(x) : \mathbb{R} \to \mathbb{R}$, and let $D : V \to V$ be the differentiation linear transformation $D(f) := f'$. For each of the following subspaces $V_i \subset V$ invariant under $D$, let $D_i : V_i \to V_i$ be the restriction of $D$ to $V_i$. Find all eigenvalues of each $D_i$.
   
   (a) $D_1$ is the span of $\{1, x\}$.
   
   (b) $D_2$ is the span of $\{2^x, 3^x\}$.
   
   (c) $D_3$ is the span of $\{\sin(x), \cos(x)\}$.

4. Let $V$ be a vector space over $\mathbb{F}$ and let $T : V \to V$ be a linear map. Fix $\lambda \in \mathbb{F}$ and $k \in \mathbb{N}$, and let $N$ be the null space of $(T - \lambda I)^k$, where $I : V \to V$ is the identity linear map $I(v) := v$ for all $v \in V$. Show that $N$ is invariant under $T$.

5. Let $V$ be a vector space over $\mathbb{C}$ and let $T : V \to V$ be a linear map. Let $N$ be a finite-dimensional subspace of $V$ invariant under $T$. Show that $N$ contains an eigenvector of $T$.

6. Let $V$ be a vector space and let $T : V \to V$ be a linear map. For any $k \in \mathbb{N}$, let $N_k$ be the null space of $T^k$ (so $N_0$ is the null space of the identity transformation).

   (a) Show that $N_k$ is invariant under $T$, for any $k$.
   
   (b) Show that $N_{k+1} \supset N_k$, for any $k$.
   
   (c) Show that if $N_{k+1} = N_k$, then $N_{k+2} = N_{k+1}$.
   
   (d) If dim$(V) = n$, show that $N_n = N_k$ for any $k \geq n$.

7. (Harder) Let $V$ be a vector space, let $T : V \to V$ be a linear map. Let $N \leq V$ be a subspace of $V$, let $W$ be the quotient space $V/N$, and let $\pi : V \to W$ be the canonical linear map given by $\pi(v) = v + N$. Show that $N$ is invariant under $T$ if and only if there exists a linear map $L : W \to W$ such that $L \circ \pi = \pi \circ T$.

8. (Harder) Find a vector space $V \neq \{0\}$ over $\mathbb{C}$ and a linear map $T : V \to V$ such that the only subspaces of $V$ invariant under $T$ are $V$ and $\{0\}$. In particular, $T$ will have no eigenvalues.

9. (Harder, and requires a little bit of field theory.) Let $M$ be an $n$-by-$n$ matrix over $\mathbb{C}$, and let $u_1, u_2, \ldots, u_k$ be algebraically independent over $\mathbb{Q}[[m_{ij} : 1 \leq i, j \leq n]]$. Show that the polynomial obtained in the proof of Theorem 4.2 from such a vector $u := (u_1, \ldots, u_n)$ does not depend on the choice of $u$.