Recall that a Jordan block is a $k \times k$ (sub)matrix with $a_{i,i} = \lambda$ and $a_{i,i+1} = 1$ and all other $a_{ij} = 0$. Further, a matrix $A$ is in Jordan form if it is a block-diagonal matrix each of whose blocks is a Jordan block.

1. (a) Find a matrix $A$ in Jordan form such that 17 is the only eigenvalue of $A$, and the dimensions of null spaces of powers of $B := A - 17I$ are as follows:

$$\dim(N_B) = 3, \dim(N_{B^2}) = 6, \dim(N_{B^3}) = 8, \dim(N_{B^4}) = 9, \dim(N_{B^5}) = \dim(N_{B^6}) = 10$$

(b) Find another matrix $C$ satisfying the same requirements.

2. (Use software! Beware of rounding errors!) Let $A := \begin{bmatrix} 5 & 0.7 & 0 & -2 & 0 \\ 1 & 3.7 & -2 & 0 & 1 \\ 2 & 0 & 2.7 & -1 & 0 \\ 2 & 0 & -1 & 2.7 & 0 \\ 2 & 0 & 0 & 0 & -1 \end{bmatrix}$. 

(a) Find the unique eigenvalue $\lambda$ of $A$.

(b) Compute the powers $B^k$ of $B := A - \lambda I$ for $k = 0, 1, 2, 3, 4$.

(c) Find a basis of $\mathbb{C}^4$ consisting of Jordan Chains of $A$.

(d) Find the change-of-basis matrix $U$ for which $U^{-1}AU$ is in Jordan form.

3. Suppose that $A$ and $B$ are similar matrices: that is, they represent the same linear transformation with respect to different bases. What can you say about $(A - \lambda I)$ and $(B - \lambda I)$ for a scalar $\lambda$?

4. Show that the following are equivalent:

(a) The unique eigenvalue of $A$ is $\lambda$.

(b) The unique eigenvalue of $(A - \lambda I)$ is 0.

5. Show that the following are equivalent for a matrix $M \in \mathbb{C}^{n \times n}$:

(a) The unique eigenvalue of $M$ is 0.

(b) $M^m$ is the zero matrix for some $m \in \mathbb{N}$.

(c) $M^n$ is the zero matrix.

Find a matrix $N \in \mathbb{C}^{5 \times 5}$ such that $N^5 = 0 \neq N^4$. 

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6. Without using Theorem 4.14 from Dym, show that the following are equivalent:

(a) For any square matrix $A$ over $C$, if 0 is the unique eigenvalue of $A$, then $A$ is similar to a matrix in Jordan form.

(b) For any square matrix $A$ over $C$, if $\lambda$ is the unique eigenvalue of $A$, then $A$ is similar to a matrix in Jordan form.

(c) Any square matrix $A$ over $C$ is similar to a matrix in Jordan form. 
   (Hint: use Theorem 4.12.)

7. Let $V$ be a vector space.

(a) Suppose that $S \subseteq V$ is linearly independent, and $w \in V$. Show that $S \cup \{w\}$ is linearly dependent if and only if $w$ is in the span of $S$.

(b) Suppose that $S$ and $T$ are subsets of $V$, and $T \subseteq \text{Span}(S)$. Show that $\text{Span}(T) \subseteq \text{Span}(S)$.

(c) Show that any linearly independent subset of $V$ is contained in a basis of $V$; if you are comfortable with Zorn’s Lemma (or transfinite induction), do not assume that $V$ is finite-dimensional.

(d) Suppose that $\{v_1, v_2, \ldots, v_m\}$ and $\{w_1, w_2, \ldots, w_n\}$ are two bases of $V$. Show that there exists some $k$ such that $B := \{v_1, \ldots, v_{m-1}, w_k\}$ is linearly independent; and that $B$ is also a basis of $V$.

(e) Use part (d) to prove that (finite) dimension is well-defined: that is, any two finite bases of the same vector space have the same size.