Math A4600: Linear Algebra

Thanksgiving for free online matrix calculators.

Thanks are due to:
- [http://www.arndt-bruenner.de/mathe/scripts/engl_eigenwert.htm](http://www.arndt-bruenner.de/mathe/scripts/engl_eigenwert.htm) that checked my eigenvalue/eigenvector computations;
- [http://matrix.reshish.com](http://matrix.reshish.com) that found null spaces for me; and
- [http://www.bluebit.gr/matrix-calculator/](http://www.bluebit.gr/matrix-calculator/) that multiplied matrices and found determinants for me, and allowed me to paste in matrices instead of retyping them.

The matrix $A$ below has only one eigenvalue, 4. To analyze it, we compute powers of $B := A - 4I$.

$$A := \begin{bmatrix} 4 & -1 & 1 & -1 & -1 \\ 0 & 3 & -1 & 1 & -1 \\ 0 & 1 & 3 & 1 & 1 \\ 2 & 1 & 1 & 5 & 1 \\ 0 & 1 & 1 & -1 & 5 \end{bmatrix} \quad A - 4I : = B = \begin{bmatrix} 0 & -1 & 1 & -1 & -1 \\ 0 & -1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 & 1 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} -2 & 0 & -2 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 \end{bmatrix} \quad B^3 = 0$$

So, the null spaces are:
- $N_{B^3}$ is everything, dimension 5.
- $N_{B^2} = \{(a, b, -a, d, e) : a, b, d, e \in \mathbb{F}\}$, dimension 4
- $N_B = \{(-s, -t, s, s, t) : s, t \in \mathbb{F}\}$, dimension 2

So, $A$ has 2 eigenvectors, so 2 Jordan chains, of lengths 3 and 2. Start with the Jordan chain of length 3.

We need $u_3 \in N_{B^3}, \notin N_{B^2}$. We can take

$$u_3 = (1, 0, 0, 0, 0)^T$$

[[or take $v_3 = (0, 0, 1, 0, 0)^T$, but not both]]

$$u_2 = Bu_3 = (0, 0, 2, 0, 0)^T$$ first column of $B$

[[ $v_2 = (1, -1, -1, 1, 1)^T$, third column of $B$]]

$$u_1 = Bu_2 = 2(-1, 1, 1, 1, -1)^T = (-2, 2, 2, 2, -2)^T$$

[[$v_1 = (-2, 2, 2, 2, -2)^T$]]

Now for the Jordan chain of length 2, need $w_2 \in N_{B^2}$ that is linearly independent from $u_2$ over $N_B$. That is, we need $a, b, d, e \in \mathbb{F}$ such that
• For any $x, y \in \mathbb{F}$, if $x(a, b, -a, d, e)^T + y(0, 0, 0, 2, 0)^T \in N_B$, then $x = y = 0$.

• Equivalently, for any $x, y, s, t \in \mathbb{F}$, if $x(a, b, -a, d, e)^T + y(0, 0, 0, 2, 0)^T = (-s, -t, s, s, t)^T$, then $x = y = 0$.

• Equivalently, for any $x, y \in \mathbb{F}$, if $xb = -xe$ and $-xa = xd + 2y$, then $x = y = 0$. (These equations come from matching $t$ and $-t$ entries, and matching the two $s$ entries: the other $s$-related equation, connecting first and third coordinates, is always true.)

Clearly, $b \neq -e$ is necessary and sufficient. So let

$$w_2 = (0, 1, 0, 0)^T$$

[[ $(1, 1, -1, 7, 17)^T$ also works, but is harder to work with.]]

(( what’s bad about $z_2 = u_2 + (-1, -1, 1, 1, 1)^T = (-1, -1, 1, 3, 1)^T$? ))

So $w_1 = (-1, -1, 1, 1, 1)$, nonzero and in $N_B$ as wanted.

$((z_1 = Bz_2 = (-2, 2, 2, 2, -2)^T))$

Check that $\{u_1, u_2, u_3, w_1, w_2\}$ is a basis: compute determinant of

$$\begin{vmatrix}
1 & 0 & -2 & 0 & -1 \\
0 & 0 & 2 & 1 & -1 \\
0 & 0 & 2 & 0 & 1 \\
0 & 2 & 2 & 0 & 1 \\
0 & 0 & -2 & 0 & 1 \\
\end{vmatrix}$$

((see that $\{u_1, u_2, u_3, z_1, z_2\}$ is not a basis: $u_1 = z_1$.))

Now

$$B : u_3 \mapsto u_2 \mapsto u_1 \mapsto 0 \text{ and } w_2 \mapsto w_1 \mapsto 0$$

$$A : u_3 \mapsto 4u_3 + u_2, u_2 \mapsto 4u_2 + u_1, u_1 \mapsto 4u_1 \text{ and } w_2 \mapsto 4w_2 + w_1, w_1 \mapsto 4w_1$$

So, with respect to this basis, the linear transformation represented by $A$ with respect to the standard basis is represented by a block-diagonal matrix with Jordan blocks:

$$\begin{bmatrix}
4 & 1 & 0 & 0 & 0 \\
0 & 4 & 1 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 4 & 1 \\
0 & 0 & 0 & 0 & 4 \\
\end{bmatrix}$$