1 introduction

The goal is to prove the Compactness Theorem of First Order Logic:

**Theorem 1.1.** Every finitely satisfiable set of first-order sentences is satisfiable.

**Definition 1.1.** A set $S$ of $L$-sentences is called finitely satisfiable if for any finite subset $S_0 \subset S$ there is an $L$-structure satisfying all formulae in $S_0$.

We start with a first-order language $L_0$ and a finitely satisfiable set $S_0$ of $L_0$-sentences; we grow the language to $L_\omega$ by adding many new constant symbols, and grow the set $S_0$ to $\Sigma$ with two extra properties: still finitely satisfiable, $\Sigma$ is also complete, and has constant witnesses. We then build an $L_\omega$-structure $U$ out of the constants of $L_\omega$, and show that it satisfies all sentences in $\Sigma$. Then the reduct of $U$ to $L_0$ satisfies all sentences in $S$, and we are done.

2 the project

1. Given a first-order language $L$ and a finitely satisfiable set $T$ of $L$-sentences, let $L'$ be a new language with lots of extra constant symbols, and $T'$ be a set of $L'$-sentences as follows:

\[ L' := L \cup \{ C_\phi \mid \phi(x) \text{ is an } L\text{-formula} \} \]
\[ T' := T \cup \{ (\exists x \phi) \rightarrow \phi(C_\phi) \mid \phi(x) \text{ is an } L\text{-formula} \} \]

Show that $T'$ is finitely satisfiable.

2. Given a first-order language $L_0$ and a finitely satisfiable set $S_0$ of $L_0$-sentences, use the definitions above to define $L_n$ and $S_n$ inductively as follows:

\[ L_{n+1} := L'_n \text{ and } S_{n+1} := S'_n \]

Let

\[ L_\omega := \cup_{n \in \mathbb{N}} L_n \text{ and } S_\omega := \cup_{n \in \mathbb{N}} S_n \]

(a) Show that for all $n$, $S_n$ is finitely satisfiable.

(b) Show that $S_\omega$ is finitely satisfiable.

(c) Show that for any $L_\omega$-formula $\phi(x)$, there is a constant symbol $C_\phi$ in $L_\omega$ such that the sentence $(\exists x \phi) \rightarrow \phi(C_\phi)$ is in $S_\omega$.

**Definition 2.1.** A set $T$ of $L$-sentences is said to have constant witnesses if for every $L$-formula $\phi$ there is a constant symbol $C_\phi$ in $L$ such that the sentence $(\exists x \phi) \rightarrow \phi(C_\phi)$ is in $T$.

3. (a) Suppose that you have countably many symbols; show that there are only countable many finite strings of these symbols. Conclude that if a first-order language $L$ has countably many non-logical symbols, then there are countably many $L$-formulae.
(b) Show that if there are countably many symbols in $L$, and countably many formulae in $L$, then there are countably many formulae in $L'$ defined above.

(c) Given countable first-order languages $L_n$ for $n \in \mathbb{N}$ such that $L_i \subset L_j$ for all $i \leq j$, show that $\cup_{n \in \mathbb{N}} L_n$ is countable. Conclude that $L_\omega$ defined above has countably many formulae.

4. Show that if a set $T$ of $L$-sentences is finitely satisfiable, and $\theta$ is an $L$-sentence, then either $T \cup \{ \theta \}$ or $T \cup \{ \neg \theta \}$ is finitely satisfiable.

5. Show that if a set $S_\omega$ of $L_\omega$-sentences is finitely satisfiable, then there exists a finitely satisfiable set $\Sigma$ of $L_\omega$-sentences such that $S_\omega \subset \Sigma$ and for every $L_\omega$-sentence $\theta$, either $\theta \in \Sigma$ or $\neg \theta \in \Sigma$.

**Definition 2.2.** A set $T$ of $L$-sentences is called complete is for every $L$-sentence $\theta$, either $\theta \in T$ or $\neg \theta \in T$.

6. So, we now have a set of $L_\omega$-sentences $\Sigma \supset S$, which is complete, finitely satisfiable, and has constant witnesses.

**Definition 2.3.** An $L$-sentence $\psi$ is a semantic consequence of a set $\Sigma$ of $L$-sentences if every $L$-structure $A$ that satisfied all sentences in $\Sigma$ also satisfies $\psi$.

7. Suppose that a set $T$ of $L$-sentences is complete and finitely satisfiable, and that an $L$-sentence $\psi$ is a semantic consequence of a finite subset of $T$; show that $\psi \in T$.

8. Suppose that a set $T$ of $L$-sentences is complete and finitely satisfiable; define two constant symbols $C$ and $D$ to be $T$-equivalent if the sentence $C = D$ is in $T$. Show that this is an equivalence relation.

9. Suppose that a set $T$ of $L$-sentences is complete and finitely satisfiable; show that if $C_i$ is $T$-equivalent to $D_i$ for all $i \leq n$, and $R$ is an $n$-ary relation symbol in $L$, then the sentence $R(C_1, \ldots, C_n)$ is in $T$ if and only if the sentence $R(D_1, \ldots, D_n)$ is in $T$.

10. Suppose that a set $T$ of $L$-sentences is complete and finitely satisfiable, and has constant witnesses. Show that for any constant symbols $C_i$ in $L$ and any $n$-ary function symbol $f$ in $L$, there is a constant symbol $C$ in $L$ such that the sentence $f(C_1, \ldots, C_n) = C$ is in $T$.

11. Suppose that a set $T$ of $L$-sentences is complete and finitely satisfiable; $C_i$ is $T$-equivalent to $D_i$ for all $i \leq n$; $f$ is an $n$-ary function symbol in $L$; and sentences $f(C_1, \ldots, C_n) = C$ and $f(D_1, \ldots, D_n) = D$ are in $T$; show that $C$ is $T$-equivalent to $D$.

12. We now construct an $L_\omega$-structure $U$ as follows: the universe $U$ will consist of $\Sigma$-equivalence classes $u_C$ of $L_\omega$ constant symbols $C$; we interpret $L_\omega$ as follows:
   
   - for a constant symbol $C$, we let $C^U := u_C$
   - for an $n$-ary relation symbol $R$,
     \[ R^U := \{(u_{C_1}, u_{C_2}, \ldots, u_{C_n}) \mid R(C_1, C_2, \ldots, C_n) \in \Sigma\} \]
   - for an $n$-ary function symbol $f$ and an $n$-tuple $(u_{C_1}, u_{C_2}, \ldots, u_{C_n})$ of elements in $U$, we let $f^U(u_{C_1}, u_{C_2}, \ldots, u_{C_n}) := u_C$ for some constant symbol $C$ in $L_\omega$ such that $f(C_1, C_2, \ldots, C_n) = C$ is in $\Sigma$.

   Verify that the functions $f^U$ are well-defined.

13. Suppose that a set $T$ of $L$-sentences is complete and finitely satisfiable, and that the $L$-sentence $\forall x \phi(x)$ is in $T$, and that $C$ is a constant symbol in $L$. Show that the $L$-sentence $\phi(C)$ is in $T$.

14. Show that $U$ satisfies $\Sigma$. Hint: induct on the complexity of a sentence $\theta$ to show that $U \models \theta$ if and only if $\theta \in \Sigma$. Further hint: use the fact that $\Sigma$ has constant witnesses to deal with the quantifier induction step.

15. Yay, we are done!