1. Let $S := \{+, \times, 0, 1\}$ be the signature of rings, let $\mathcal{N}$ be the $S$-structure with universe $\mathbb{N}$ and usual interpretations of the symbols in $S$, and let $t_n$ be $S$-terms with no variables such that $t_n^\mathcal{N} = n$. Let $\mathcal{A}$ be an $S$-structure and $\alpha$ an $\mathcal{A}$-assignment satisfying $Th(\mathcal{N}) \cup \{x \neq t_n \mid n \in \mathbb{N}\}$.

(a) Show that such a structure $\mathcal{A}$ exists, using the Compactness of First-Order Logic; as we did in class.

(b) Define $h : \mathbb{N} \to \mathcal{A}$ by $h(n) := t^n_\mathcal{A}$. Show that $h$ is an $S$-embedding. These elements $t^n_\mathcal{A}$ are the standard elements of $\mathcal{A}$; others are non-standard.

(c) Find an $S$-formula $\phi_\leq$ such that for any $\mathcal{N}$-assignment $\beta$, $\Vdash_\mathcal{N} \phi_\leq[\beta]$ if and only if $\beta(v_1) \leq \beta(v_2)$. Show that $\phi_\leq$ also defines an ordering on the universe $A$ of $\mathcal{A}$.

(d) Are any nonstandard elements of $\mathcal{A}$ less than $0^\mathcal{A}$ in the sense of the ordering given by $\phi_\leq$? Between $t_3^\mathcal{A}$ and $t_4^\mathcal{A}$?

(e) Find an $S$-formula $\eta$ such that for any $\mathcal{N}$-assignment $\beta$, $\Vdash_\mathcal{N} \eta[\beta]$ if and only if $\beta(v_1)$ is an even number. Show that there is an $\mathcal{A}$-assignment $\alpha$ such that $\Vdash_\mathcal{A} \eta[\alpha]$ and $\alpha(v_1)$ is non-standard. For what other properties of natural numbers (besides “even”) can you do the same?

2. Use the Compactness theorem for first-order logic to show that the class of connected graphs is not axiomatizable. That is, there is no signature $S$ and no set of $S$-sentences $T$ such that every model of $T$ is a connected graph and every connected graph is a model of $T$.

3. Let $S := \{+, \times, 0, 1\}$ be the signature of rings.

(a) Write down a set $T$ of $S$-sentences whose models are algebraically closed fields. (That is, all models of $T$ are algebraically closed fields, and all algebraically closed fields are models of $T$.)

(b) Use the Compactness theorem for first-order logic to show that there is no set $T'$ of $S$-sentences such that its models are non-algebraically closed fields.

4. Let $S := \emptyset$ be the the first-order signature with no non-logical symbols.

(a) Give an explicit enumeration $\{\theta_n \mid n \in \mathbb{N}\}$ of all $S$-sentences. In your enumeration, what is $\theta_0$? $\theta_1$? $\theta_{17}$?

(b) What $S$-structure will be produced by the Henkin construction starting with $\Gamma := \emptyset$ with your enumeration?

5. Ask an interesting question about first-order logic and try to answer it. This question is as serious as the rest of them!