Math A4400: Mathematical Logic

7th problem set, due at 2pm on Wednesday, November 20th.

Bring your solutions class, or slide them under the door of my office NAC 6278.

1. Let \( S \) be a signature with one ternary relation symbol \( R \) and no other symbols. For each of the following four \( S \)-structures \( A_i \), find an \( S \)-sentence \( \theta_i \) such that \( \models_{A_i} \theta_i \) and \( \not\models_{A_j} \theta_i \) for any \( j \neq i \).
   
   (a) \( A_1 := \mathbb{Q} \) and \( R^{A_1} := \{(a,b,c) \in \mathbb{Q}^3 \mid a + b = c \} \).
   
   (b) \( A_2 := \mathbb{Q} \) and \( R^{A_2} := \{(a,b,c) \in \mathbb{Q}^3 \mid a \cdot b = c \} \).
   
   (c) \( A_3 := \mathbb{Q} \) and \( R^{A_3} := \{(a,b,c) \in \mathbb{Q}^3 \mid a \leq b \leq c \} \).
   
   (d) \( A_4 := \mathbb{Z} \) and \( R^{A_4} := \{(a,b,c) \in \mathbb{Z}^3 \mid a \leq b \leq c \} \).

2. Let \( S \) be the signature with one binary relation \( Q \), and let \( A \) be the \( S \)-structure with universe \( \{a \in \mathbb{Z} \mid a \geq 17\} \) and \( Q^A := \{(a,b) \in \mathbb{Z}^2 \mid a < b \} \). Let \( \theta \) be the \( S \)-formula \( Qv_1v_2 \land \neg \exists v_3 (Qv_1v_3 \land Qv_3v_2) \). For each of the formulas \( \phi_i \) below, find an \( A \)-assignment \( \alpha_i \) such that \( \models_A \phi_i[\alpha_i] \) and \( \not\models_A \phi_j[\alpha_i] \) for any \( j \neq i \).
   
   (a) \( \phi_1 := \forall v_3 (Qv_3v_7 \rightarrow v_5v_7) \).
   
   (b) \( \phi_2 := \exists v_3 (Qv_3v_7 \land (\theta v_3(v_3)) \land \varphi_2(v_7)) \).
   
   (c) \( \phi_3 := \neg \forall v_4 (Qv_4v_7 \rightarrow \theta v_4(v_4)) \).
   
   (d) \( \phi_4 := \exists v_3 \exists v_6 (Qv_3v_6 \land Qv_6v_7 \land \forall v_8 (Qv_8v_7 \rightarrow (\neg v_8v_5 \lor v_8v_6))) \).

3. Let \( S \) be the signature with one binary function symbol \( \times \), and let \( A \) be the \( S \)-structure with universe \( \mathbb{Q} \) and with \( \times^A(a,b) := a \cdot b \). Show that there is no \( S \)-formula \( \phi \) such that \( \models_A \phi[\alpha] \) if and only if \( \alpha(v_1) \cdot \alpha(v_2) = \alpha(v_3) \).

   Hint: consider automorphisms of \( A \).

4. (The last problem from the second midterm.) Let \( S \) be a signature with a binary function symbol \( F \). Let \( A \) be the \( S \)-structure with universe \( \mathbb{Z} \) and \( F^A(a,b) := a + b \). Let \( B := \{-1,1\} \), and let \( h : A \rightarrow B \) be given by

   \[
   h(n) := \begin{cases} 
   1 & \text{if } n \geq 0 \\
   -1 & \text{if } n < 0 
   \end{cases}
   \]

   If there is an \( S \)-structure \( B \) with universe \( B \) for which \( h \) is an \( S \)-homomorphism, determine \( F^B \). Otherwise, prove that there is no such \( S \)-structure

5. Ask an interesting question about the material we covered since the midterm and try to answer it. This question is as serious as the rest of them!