1. Suppose that $X$, $Y$, and $Z$ are symbols in a signature $S$.

   (a) Could both $XYZ$ and $XZY$ be $S$-formulas? If so, what kinds of symbols would $X$, $Y$ and $Z$ have to be?

   (b) Could both $XYZ$ and $ZYX$ be $S$-terms? If so, what kinds of symbols would $X$, $Y$ and $Z$ have to be?

   List all possibilities and carefully prove your answers.

2. Let $S$ be a signature with one binary function symbol $F$ and no other symbols. Let $\phi := \forall x Fxy = x$ and $\psi := \forall y Fxy = x$ be two $S$-formulas. Consider the following three $S$-structures with universe $\mathbb{Z}$:

   - $F^T(a, b) := a \cdot b$
   - $F^P(a, b) := a + b$
   - $F^F(a, b) := a$

   For each of the six possible combinations of formula and structure, describe the set of assignments that satisfy that formula in that structure.

3. Let $S$ be a signature, $x$ a variable, and $\tau$ an $S$-term. Prove or refute the following.

   (a) If $t$ is another $S$-term, and $t'$ is the expression obtained by replacing every instance of $x$ in $t$ with $\tau$, then $t'$ is also an $S$-term.

   (b) If $\alpha$ is an $S$-formula, and $\beta$ is the expression obtained by replacing every instance of $x$ in $\alpha$ with $\tau$, then $\beta$ is also an $S$-formula.

4. Ask an interesting question about this week's material and try to answer it. This question is as serious as the rest of them!

   **Bonus** Let $S$ be the signature with two binary function symbols $+$ and $\times$ and two constant symbols $0$ and $1$. Let $\mathcal{N}$ be the $S$-structure with universe $\mathbb{N}$ and the usual interpretations of the symbols of $S$.

   (a) Show that for every natural number $n$, there is an $S$-term $n \cdot \alpha$ with no variables such that $\mathcal{N} \cdot [\alpha] = n$ for any $\mathcal{N}$-assignment $\alpha$.

   (b) Suppose that $\mathcal{A}$ is another $S$-structure, and suppose that for any atomic $S$-formula $\phi$ with no variables, any $\mathcal{A}$-assignment $\alpha$, and any $\mathcal{N}$-assignment $\nu$, $\mathcal{A} \cdot [\alpha] = \mathcal{N} \cdot [\phi[\nu]]$ if and only if $\mathcal{A} \cdot \phi[\alpha] = \mathcal{N} \cdot \phi[\nu]$. Show that there exists a unique $S$-embedding from $\mathcal{N}$ to $\mathcal{A}$.