Standard bases in differential algebra

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April 7, 2006

Gröbner bases of polynomial ideals

Let $R = k[x_0, \dots, x_m]$ be the polynomial ring over a field k. By $T = T(X)$ we denote the semigroup of monomials generated by elements of $X = \{x_0, \dots, x_m\}$. Then, T forms a basis of R; i.e., any $a \in R$ may be represented as a finite linear combination of monomials with nonzero coefficients from k, and this representation is *unique*.

Admissible monomial orderings

Suppose that the monomials are ordered so that $\forall\ \theta \in T$

$$1 \preceq \theta, \tag{1}$$

$$\theta_1 \prec \theta_2 \implies \theta\theta_1 \prec \theta\theta_2. \tag{2}$$

For $\theta = x_0^{e_0} \dots x_m^{e_m}$, define the degree as $\deg \theta = e_0 + \cdots + e_m$.

Example 1 *(ordering* `lex`*).* Let $\theta_1 = x_0^{e_0} \dots x_m^{e_m}$, $\theta_2 = x_0^{i_0} \dots x_m^{i_m}$. Then, $\theta_1 \prec_{\text{lex}} \theta_2$ if either $e_0 < i_0$ or $e_j = i_j$ for $j = 0, \dots, k$ and $e_{k+1} < i_{k+1}$ for some k $(0 < k < m)$.

Example 2 *(ordering* `deglex`*).* We set $\theta_1 = x_0^{e_0} \dots x_m^{e_m} \prec_{\text{deglex}} \theta_2 = x_0^{i_0} \dots x_m^{i_m}$ if either $\deg \theta_1 < \deg \theta_2$ or $\deg \theta_1 = \deg \theta_2$ and $\theta_1 \prec_{\text{lex}} \theta_2$.

Example 3 *(ordering* `degrevlex`*).* Let $\theta_1 = x_0^{e_0} \dots x_k^{e_k}$, $\theta_2 = x_0^{i_0} \dots x_k^{i_k}$. We set $\theta_1 \prec_{\text{degrevlex}} \theta_2$ if $\deg \theta_1 < \deg \theta_2$ or $\deg \theta_1 = \deg \theta_2$ and $e_j = i_j$ for $j = k+1, \dots, m$ and $e_k < i_k$ for some $1 < k \leq m$.

It is well known that any monomial ordering can be specified by an $r \times (m+1)$ *monomial matrix* $\mathcal{M}$ with real entries and lexicographically positive columns such that $\operatorname{Ker}_{\mathbb{Q}} \mathcal{M} = \{0\}$:

$$\mathcal{M}\begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_m \end{pmatrix} \prec_{\mathrm{lex}} \mathcal{M}\begin{pmatrix} \beta_0 \\ \vdots \\ \beta_m \end{pmatrix} \iff x_0^{\alpha_0} \dots x_m^{\alpha_m} \prec x_0^{\beta_0} \dots x_m^{\beta_m}.$$

Lex

$$\begin{pmatrix} 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ldots$$

DegLex

$$\begin{pmatrix} 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ldots$$

DegRevLex

$$\begin{pmatrix} 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \ldots$$

WtRevLex

$$\begin{pmatrix} 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \ldots$$

A monomial ordering given, we can distinguish the *leading monomial* $\mathrm{lm}(a)$ in any polynomial $a \in R$. A set G of generators of an ideal $I \subset k[x_0, \ldots, x_m]$ is a *Gröbner basis* of I if the monomial ideal generated by $\{\mathrm{lm}(g) \forall g \in G\}$ coincides with the monomial ideal generated by $\{\mathrm{lm}(g) \forall g \in I\}$. This means that, for any $f \in I$, there exists $g \in G$ (maybe not unique!) such that $\mathrm{lm}(g)$ divides $\mathrm{lm}(f)$.

A polynomial f is reduced with respect to a polynomial g if no monomial present in f with nonzero coefficient is divisible by the leading monomial of g. A set $G \subset k[x_0, \ldots, x_m]$ is autoreduced if, for any distinct $g_1, g_2 \in G$, polynomial g_1 is reduced with respect to g_2 (and vice versa). Any autoreduced set is finite. Any admissible monomial ordering induces a partial order on the set of autoreduced sets. The minimal element in the set of all autoreduced subsets of an ideal $I \subset k[x_0, \ldots, x_m]$ is a Gröbner basis of I. Without loss of generality, we may consider only monic autoreduced sets (such that leading coefficient of their elements are equal to 1).

For any ideal $I = (g_1, \dots, g_t)$, the monic autoreduced Gröbner basis is unique and can be found in a finite number of steps.

Differential algebra

Let R be a unitary commutative domain. A mapping $\delta : R \mapsto R$ such that

$$\delta(a + b) = \delta a + \delta b$$

$$\delta(ab) = \delta a \cdot b + a \cdot \delta b$$

for any $a, b \in R$ is called a *derivation operator* or *differentiation* of R.

An ordinary differential ring (field) is a ring (field) with a derivation operator δ. A ring (field) with several pairwise commuting derivation operators is called a *partial differential ring (field)*.

Example 4.

1. Any ring R can be treated as a differential ring with zero differentiation.
2. The ring of infinitely differentiable (with respect to d/dx) functions on an interval is an ordinary differential ring.
3. Let $D[x]$ be the ring of polynomials in x over a ring D. For any $p(x) \in D[x]$, there exists a unique derivation operator δ on $D[x]$ such that $\delta a = 0$ for any $a \in D$ and $\delta x = p(x)$.

Let R be a differential ring (field) with a derivation operator δ. The set $c \in R : \delta c = 0$ is a subring (subfield) of constants of R. The element δa is referred to as the *derivative* of $a \in R$ and is often denoted by a'. The element $\delta^n(a)$ is the nth derivative of a (denoted by $a^{(n)}$).

Let R be a differential domain and $\Delta = \{d_1, \ldots, d_m\}$ the basic set of the derivation operators on R. The noncommutative ring $D = R[d_1, \ldots, d_m]$ of skew polynomials with coefficients in R and the commutation rules $d_i d_j = d_j d_i$, $d_i a = a d_i + d_i(a)$ for any $a \in R$, $d_i, d_j \in \Delta$, is called the *ring of (linear) differential operators*. If the derivation operators are trivial on R, then D is isomorphic to the ring of commutative polynomials in the same generators.

The ring of linear differential operators is used for investigating algebraic properties of systems of linear partial differential equations. The theory of Gröbner bases can be directly applied to left (right) ideal in such rings and to left (right) modules over these rings.

Let R be a (partial) differential ring with derivation operators $\Delta = \{d_1, \ldots, d_m\}$. We denote $T = \bigcup_{i=1}^{\infty} T_i$, where $T_i = \{d_1^{i_1} \ldots d_m^{i_m} \mid i = i_1 + \cdots + i_m,\ i_1 \geq 0, \ldots, i_m \geq 0, \}$.

The ring of polynomials $R\{y\} = R\{y_1, \ldots, y_n\}$ in infinitely many variables $\{\theta y_j\}_{\theta \in T,\ 1 \leq j \leq n}$ over a differential ring R is referred to as the *ring of differential polynomials* over R. Its elements are *differential polynomials*. The derivation operators from Δ act on coefficients of a differential polynomial by definition of the differential ring R and on the generators θy_j by the rule: $d(\theta y_j) = (d\theta) y_j$. We define the *degree*, $\deg f$, of a differential polynomial $f \in R\{y_1, \ldots, y_n\}$ as the degree of f as a polynomial in variables $\{\theta y_j\}_{\theta \in T,\ 1 \leq j \leq n}$ and its *order*, $\operatorname{ord} f$, is the maximal order of derivatives present in f:

$$\operatorname{ord} f = \min(i \mid f \in R[\theta y_j]_{\theta \in T_i,\ 1 \leq j \leq n}).$$

A differential ideal can have no finite system of differential generators.

Example 5. Let $\mathcal{F}$ be an ordinary differential field and $\mathcal{F}\{y\}$, the ring of differential polynomials in differential variable y. Then, the sequence of differential ideals

$$[y^2] \subset [y^2, (dy)^2] \subset \cdots \subset [y^2, \ldots, (d^i y)^2] \subset \cdots \subset \mathcal{F}\{y\}$$

is an infinite strictly increasing sequence.

A differential ring R is a Ritt algebra if R contains the field of rational numbers $\mathbb{Q}$. An ideal I is perfect (radical) if $a^n \in I \implies a \in I$.

Theorem 1 (Ritt–Raudenbush theorem). *If a Ritt algebra R satisfies the ascending chains condition for perfect (radical) differential ideals, then the differential ring $R\{x\}$ obtained by adjoining a differential variable x to R also satisfies this condition.*

Theorem 2 (Decomposition theorem). *Any perfect differential ideal in any differential ring R with ascending chains property can be represented as the intersection of a finite set of prime differential ideals (in particular, this is valid for $\mathcal{F}\{y_1, \ldots, y_n\}$, where $\mathcal{F}$ is a differential field of characteristic zero).*

Example 6. Consider the differential polynomial $A = (y')^2 + y \in \mathcal{F}\{x\}$ (F is an ordinary differential field). This polynomial is absolutely irreducible. However the differential ideal $[A]$ is not prime. It is not even perfect. In particular, one can prove that $y''' \notin [A]$, but $y''' \in \{A\}$. The radical of $[A]$ is not a prime ideal. It can be represented as $\{A\} = [y] \cap \mathfrak{p}$, where the differential ideal $\mathfrak{p}$ is defined by the condition $f \in \mathfrak{p}$ iff $f \cdot (y')^k \in [A]$ for some $k \in \mathbb{N}$.

Let $\mathcal{F}$ be a differential field, $A \in \mathcal{F}\{y_1, \ldots, y_n\}$, $A \notin \mathcal{F}$, and a ranking of $\{y_1, \ldots, y_n\}$ be given. The derivative θy_j of highest rank present in a differential polynomial A is called the *leader* of A (denoted $\mathbf{u}_A$). If $d = \deg_{\mathbf{u}_A} A$, then $A = \sum_{i=0}^{d} I_i \mathbf{u}_A^i$, where $I_0, \ldots, I_d$ are uniquely defined polynomials free of $\mathbf{u}_A$. The differential polynomial $I_A = I_d$ is the *initial* of A and $S_A = \sum_{i=1}^{d} i I_i \mathbf{u}_A^{i-1}$ is the *separant* of A.

Let $A, F \in \mathcal{F}\{y_1, \ldots, y_n\}$, $A \notin \mathcal{F}$. A differential polynomial F is *partially reduced* with respect to A, if F contains no proper derivatives $\theta \mathbf{u}_A$ of the leader of A. If F is partially reduced with respect to A and $\deg_{\mathbf{u}_A} F < \deg_{\mathbf{u}_A} A$, then F is *reduced* with respect to A.

A set $\mathcal{A}$ is autoreduced if A_i is reduced with respect to A_j for any $A_i, A_j \in \mathcal{A}$.

Any autoreduced set is finite.

Autoreduced sets can be compared as in polynomial rings. The minimal autoreduced subset of a differential ideal I is its characteristic set.

For an autoreduced set $\mathcal{A}$, we can define a pseudoreduction relation $\xrightarrow[\mathcal{A}]{}$.

If $\mathfrak{p}$ is a prime differential ideal and $\mathcal{A}$ is its characteristic set, then $a \xrightarrow[\mathcal{A}]{} 0 \iff a \in \mathfrak{p}$.

Differential ideals satisfying this property are called characterizable (depends on the ranking).

Differential G-bases by Ollivier and Carra–Ferro

A set $G \subset I$ is a differential G-basis of a differential ideal I if the leading monomials of G and their derivatives generate the set of leading monomials of I.

To use this approach we have to order the set of differential monomials and to differentiate the differential monomials (e.g., lexicographically).

The differential ideal $[y^2] \subset \mathcal{F}\{y\}$ has no differential G-basis (under the lexicographic ordering).

It has a differential G-basis (consisting of one element) for *degrevlex* ordering [Zobnin].

- An *ordinary differential ring* $\mathcal{R}$ is a commutative ring with a derivative operator δ.
- $\Theta := \{\delta^k : k \geq 0\}$.
- An ideal I of $\mathcal{R}$ is *differential* iff $\delta I \subset I$.
- $[F]$ denotes the differential ideal generated by F.
- $\mathcal{F}$ is a differential *field of constants* of characteristic zero.
- $\mathcal{F}\{y\} := \mathcal{F}[y, \delta y, \delta^2 y, \ldots]$ — a ring of differential polynomials.
- $y_i := \delta^i y$.
- $\mathbb{M}$ — the set of all differential monomials.
- $\mathrm{lm}_{\prec} f$ — the *leading monomial* of a polynomial $f \notin \mathcal{F}$ w.r.t. $\prec$.

Admissible orderings

An *admissible ordering* on the set of differential monomials $\mathbb{M}$ must satisfy the following axioms:

- $M \prec N \implies MP \prec NP \quad \forall\, M, N, P \in \mathbb{M};$
- $1 \preceq P \quad \forall\, P \in \mathbb{M};$
- $y_i \prec y_j \iff i < j.$

These properties are sufficient to guarantee that any admissible ordering well orders $\mathbb{M}$ (Zobnin, 2003).

Examples: **lex**, **deglex**, **wt-lex**, **degrevlex**, **wt-revlex**, ...

It is well known that any monomial ordering can be specified by an $m \times (k+1)$ *monomial matrix* $\mathcal{M}$ with real entries and lexicographically positive columns such that $\operatorname{Ker}_{\mathbb{Q}} \mathcal{M} = \{0\}$:

$$\mathcal{M} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_k \end{pmatrix} \prec_{\text{lex}} \mathcal{M} \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_k \end{pmatrix} \iff y_0^{\alpha_0} \ldots y_k^{\alpha_k} \prec y_0^{\beta_0} \ldots y_k^{\beta_k}.$$

Definition 1. A set of monomial matrices $\{\mathcal{M}_k\}$ is called *concordant* if the matrix $\mathcal{M}_{k-1}$ can be obtained from $\mathcal{M}_k$ by deleting the rightmost column and then by deleting a row of zeroes, if such a row exists.

Theorem. Any admissible ordering on differential monomials can be specified by a concordant set of monomial matrices or, equivalently, by an infinite monomial matrix.

Examples of admissible orderings

Lex

$$\begin{pmatrix}\mathbf{1}\end{pmatrix}, \begin{pmatrix}\mathbf{0} & \mathbf{1}\\ 1 & \mathbf{0}\end{pmatrix}, \begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{1}\\ 0 & 1 & \mathbf{0}\\ 1 & 0 & \mathbf{0}\end{pmatrix}, \begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}\\ 0 & 0 & 1 & \mathbf{0}\\ 0 & 1 & 0 & \mathbf{0}\\ 1 & 0 & 0 & \mathbf{0}\end{pmatrix} \ldots$$

$$\begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$$

DegLex

$$\begin{pmatrix}\mathbf{1}\end{pmatrix}, \begin{pmatrix}1 & \mathbf{1}\\ \mathbf{0} & \mathbf{1}\end{pmatrix}, \begin{pmatrix}1 & 1 & \mathbf{1}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}\\ 0 & 1 & \mathbf{0}\end{pmatrix}, \begin{pmatrix}1 & 1 & 1 & \mathbf{1}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}\\ 0 & 0 & 1 & \mathbf{0}\\ 0 & 1 & 0 & \mathbf{0}\end{pmatrix} \ldots$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \ldots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \end{pmatrix}$$

Examples of orderings (ctd.)

DegRevLex

$$(\mathbf{1}), \left(\begin{smallmatrix}1 & \mathbf{1}\\ \mathbf{0} & \mathbf{1}\end{smallmatrix}\right), \begin{pmatrix}1 & 1 & \mathbf{1}\\ 0 & 1 & \mathbf{1}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}\end{pmatrix}, \begin{pmatrix}1 & 1 & 1 & \mathbf{1}\\ 0 & 1 & 1 & \mathbf{1}\\ 0 & 0 & 1 & \mathbf{1}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}\end{pmatrix} \dots$$

$$\begin{pmatrix}1 & 1 & 1 & 1 & \dots\\ & 1 & 1 & 1 & \dots\\ & & 1 & 1 & \dots\\ & & & 1 & \dots\\ \dots & \dots & \dots & \dots & \dots\end{pmatrix}$$

WtRevLex

$$(\mathbf{1}), \left(\begin{smallmatrix}1 & \mathbf{2}\\ \mathbf{0} & \mathbf{1}\end{smallmatrix}\right), \begin{pmatrix}1 & 2 & \mathbf{3}\\ 0 & 1 & \mathbf{1}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & \mathbf{4}\\ 0 & 1 & 1 & \mathbf{1}\\ 0 & 0 & 1 & \mathbf{1}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}\end{pmatrix} \dots$$

$$\begin{pmatrix}1 & 2 & 3 & 4 & \dots\\ & 1 & 1 & 1 & \dots\\ & & 1 & 1 & \dots\\ & & & 1 & \dots\\ \dots & \dots & \dots & \dots & \dots\end{pmatrix}$$

δ-stability

An admissible ordering $\prec$ is called

- δ-stable, if $\boxed{M \preceq N} \quad \Rightarrow \quad \boxed{\operatorname{lm}_\prec \delta M \preceq \operatorname{lm}_\prec \delta N}$;
- strictly δ-stable, if $\boxed{M \prec N} \quad \Rightarrow \quad \boxed{\operatorname{lm}_\prec \delta M \prec \operatorname{lm}_\prec \delta N}$.

Example. **Lex** and **deglex** are strictly δ-stable.

Degrevlex and **wtrevlex** are δ-stable, but not strictly δ-stable, since $y_i^2 \succ y_{i-1}y_{i+1}$, but $\operatorname{lm} \delta y_i^2 = \operatorname{lm} \delta y_{i-1}y_{i+1}$.

$\boldsymbol{\delta}$-lexicographic and $\boldsymbol{\beta}$-orderings

For $\prec$ the following are equivalent:

- $\operatorname{lm}_\prec \delta M = \operatorname{lm}_{\text{lex}} \delta M$ for any monomial M;
- $y_i y_j \prec y_{i-1} y_{j+1}$ for all $0 < i \leq j$,
  i.e., $\prec$ is lexicographic on isobaric monomials of degree 2;

We call such orderings *δ-lexicographic*.

Example. The orderings **lex**, **deglex** and **wt-lex** are δ-lexicographic.

If, in contrast, all summands in $\delta^k M$ are compared inverse lexicographically then we call $\prec$ a *β-ordering*.

Example. **Degrevlex** and **wt-degrevlex** are β-orderings.

$\boldsymbol{\delta}$-fixedness

Definition 2. An admissible ordering $\prec$ is $\underline{\delta\text{-}\textit{fixed}}$ if

$$\forall\, f \in \mathcal{F}\{y\} \setminus \mathcal{F} \qquad \exists\, M \in \mathbb{M}; \quad \exists\, k_0, r \in \mathbb{N} :$$

$$\operatorname{lm}_{\prec} \delta^k f = M y_{r+k} \quad \text{for all} \quad k \geq k_0.$$

Example. Any δ-lexicographic ordering is δ-fixed.

Concordance with quasi-linearity

Let $\prec$ be an admissible ordering.

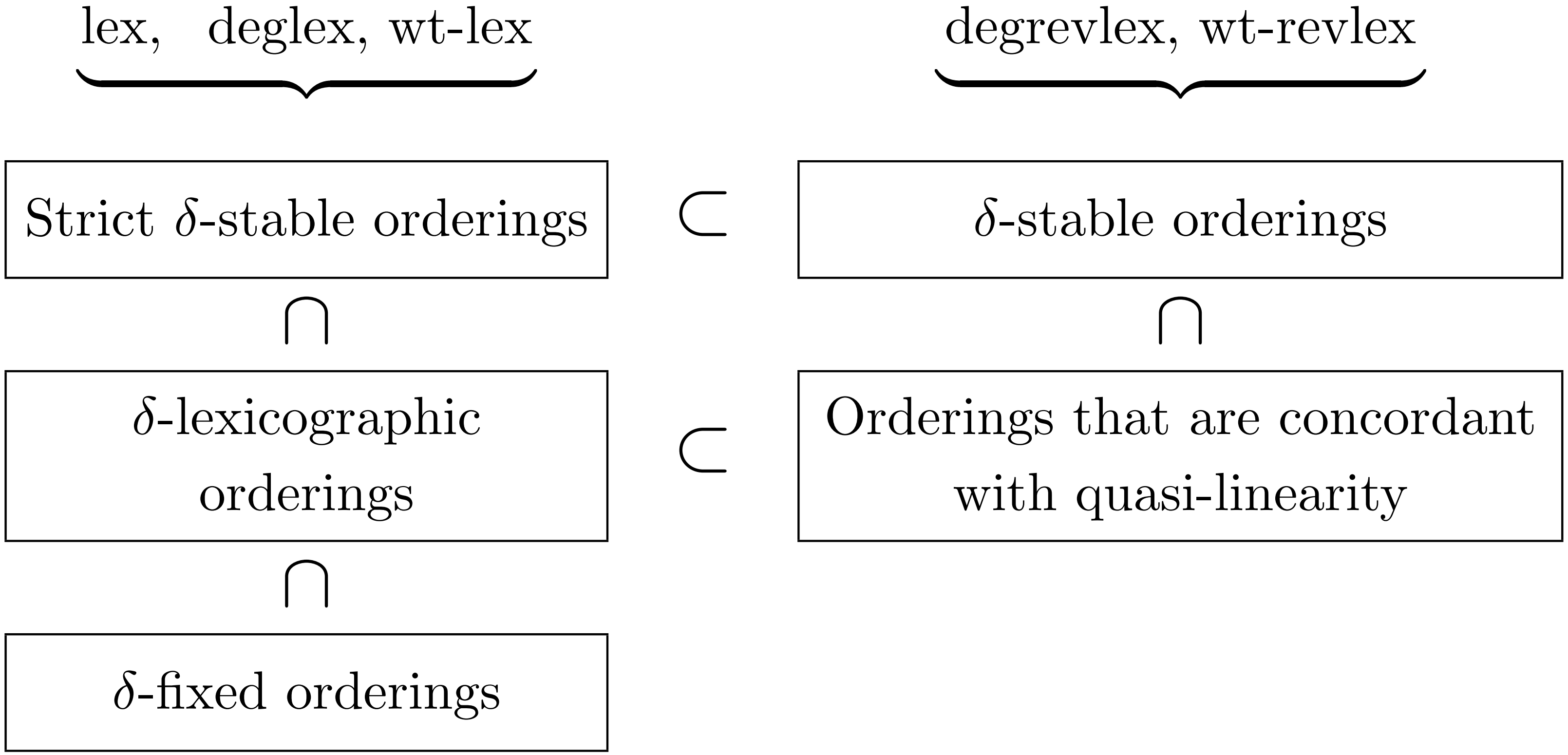
A polynomial $f \in \mathcal{F}\{x\} \setminus \mathcal{F}$ is $\prec$-*quasi-linear* if $\deg \mathrm{lm}_{\prec} f = 1$.

Example. $f = y_1 + y_0^2$ is quasi-linear w.r.t. **lex**, but not **deglex**.

We say that $\prec$ is *concordant with quasi-linearity* if the derivative of any $\prec$-quasi-linear polynomial is quasi-linear too.

Example. **Lex**, **deglex**, **degrevlex** are concordant with quasi-linearity, as well as any δ-lexicographic ordering.

Relations between orderings



$\underbrace{\text{lex, deglex, wt-lex}}$

$\underbrace{\text{degrevlex, wt-revlex}}$

Strict δ-stable orderings $\subset$ δ-stable orderings

$\cap$ $\cap$

δ-lexicographic orderings $\subset$ Orderings that are concordant with quasi-linearity

$\cap$

δ-fixed orderings

Differential standard bases

Fix an admissible ordering $\prec$. Consider a differential ideal I of $\mathcal{F}\{x\}$.

A set $G \subset I$ is a *differential standard basis* of I if ΘG is an algebraic Gröbner basis of I in $\mathcal{F}[y_0, y_1, y_2, \ldots]$ (possibly, infinite).

A DSB is *reduced* if every $g \in G$ is reduced w.r.t. $\Theta\,(G \setminus \{g\})$.

Example. Any linear ideal has a **finite** differential standard basis.

Unfortunately, differential standard bases are often **infinite**:

Example. The ideal $[y^2]$ does not have finite DSB w.r.t. **lex**.

Finiteness criterion

Let I be a proper differential ideal of $\mathcal{F}\{y\}$.

Necessary condition. For a **δ-fixed** ordering $\prec$

$$\boxed{I \text{ has a finite DSB w.r.t. } \prec} \quad \Rightarrow \quad \boxed{\begin{array}{c} I \text{ contains} \\ \text{a } \prec\text{-quasi-linear polynomial} \end{array}}.$$

Sufficient condition.

For a **concordant with quasi-linearity** ordering $\prec$

$$\boxed{I \text{ has a finite DSB w.r.t. } \prec} \quad \Leftarrow \quad \boxed{\begin{array}{c} I \text{ contains} \\ \text{a } \prec\text{-quasi-linear polynomial} \end{array}}.$$

Corollary. For **δ-lexicographic** orderings the condition is necessary and sufficient.

Corollaries

GENERALIZATIONS OF G. CARRÀ FERRO'S THEOREMS:

Corollary. Let $\prec$ be **$\boldsymbol{\delta}$-fixed**.
If the degree of each monomial in $f_1, \ldots, f_n$ is greater than 1 then $[f_1, \ldots, f_n]$ has no finite DSB w.r.t. $\prec$.

Corollary. Let $\prec$ be **strictly $\boldsymbol{\delta}$-stable**. The reduced DSB of $[f]$ w.r.t. $\prec$ consists of f itself $\iff$ f is $\prec$-quasi-linear.

KEY ROLE OF **lex**:

$$\boxed{\text{A DSB w.r.t. a } \boldsymbol{\delta}\text{-}\textbf{fixed} \text{ ordering is finite}} \Rightarrow \boxed{\text{A } \textbf{lex} \text{ DSB is also finite}}\,.$$

Improved Ollivier process

Implementation in Maple: http://shade.msu.ru/~difalg/DSB.

Input:

$F \subset \mathcal{F}\{y\}$, a finite set of polynomials;

$\prec$, a δ-fixed admissible ordering
that is concordant with quasi-linearity.

Output:

Reduced differential standard basis of $[F]$ if it is finite.

Otherwise the process does not stop.

Improved Ollivier process (ctd.)

```
$G := F$;    $H := \emptyset$;
$s := \max_{f \in F} \operatorname{ord} f$;    $k := 0$;
repeat
    $G_{old} := \emptyset$;
    while $G \neq G_{old}$ do
        $H :=$ DiffComplete $(G, s + k)$;
        $G_{old} := G$;
        $G :=$ ReducedGröbnerBasis $(H, \prec)$;
    end do;
    $k := k + 1$;
until $G \subset \mathcal{F}$ or $G$ contains a quasi-linear polynomial;
return DiffAutoreduce $(G, \prec)$;
```

Finite bases: an example

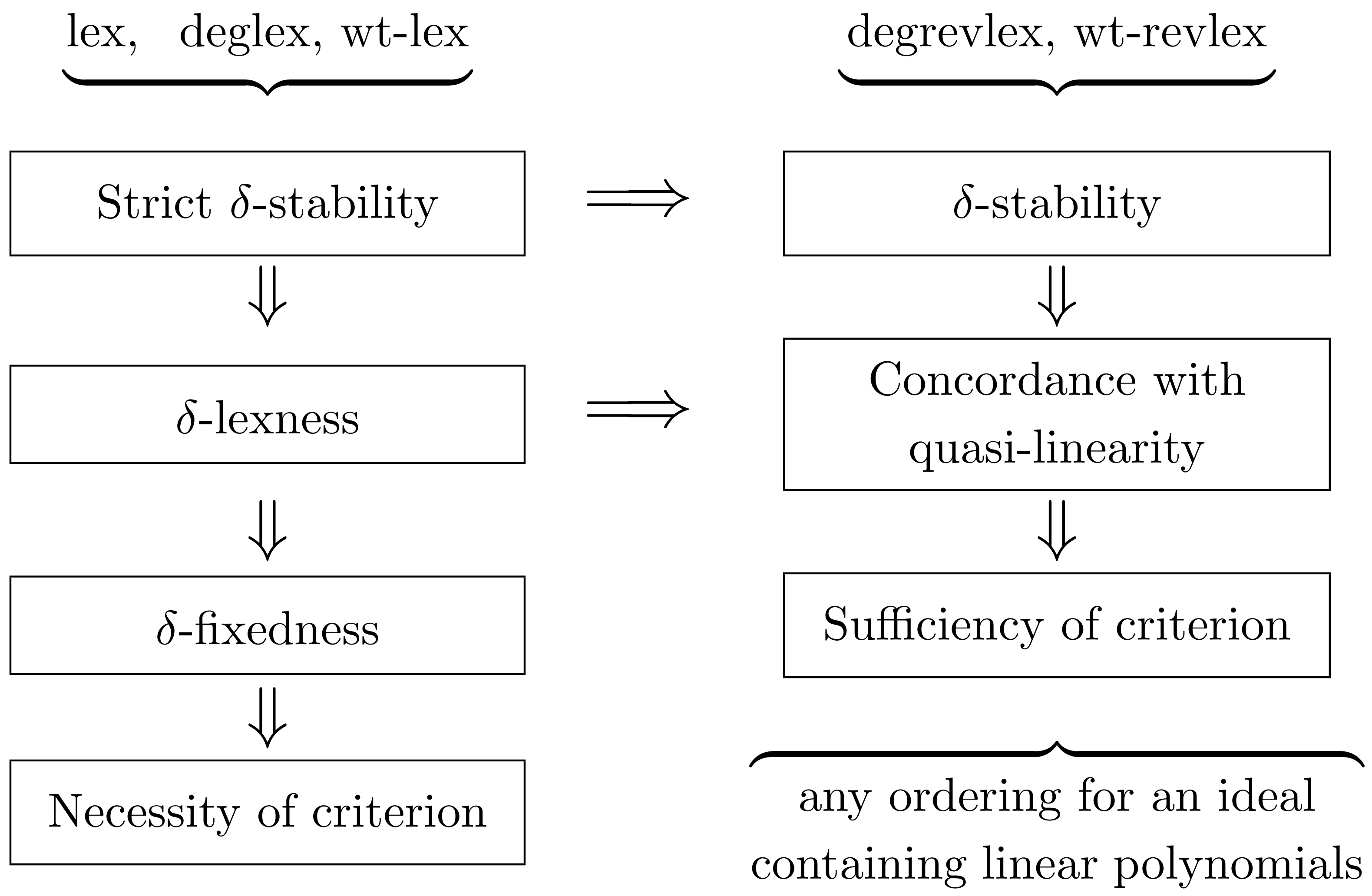
Fix the **pure lexicographic** ordering.

Consider the DSB of the ideals $[y_1^n + y]$, $n \geq 3$:

- $\boldsymbol{y_1^n} + y_0$;
- $n\, \boldsymbol{y_0}\, \boldsymbol{y_2} - y_1^2$;
- $n\, \boldsymbol{y_1^{n-2}}\, \boldsymbol{y_2^2} + y_2 = y_2\,(n\, y_1^{n-2}\, y_2 + 1)$;
- $\boldsymbol{y_3} - n(n-2)\, y_1^{n-3}\, y_2^3$.

The DSB are finite, since $[y_1^n + y]$ contains a quasi-linear polynomial.

By the way, one can prove that these ideals are radical.



lex, deglex, wt-lex

degrevlex, wt-revlex

Strict δ-stability $\Longrightarrow$ δ-stability

δ-lexness $\Longrightarrow$ Concordance with quasi-linearity

δ-fixedness

Sufficiency of criterion

Necessity of criterion

any ordering for an ideal containing linear polynomials

Finite DSB and radical ideals

Let $\prec$ be a δ-fixed and concordant with quasi-linearity ordering.

Theorem (M. V. Kondratieva, A. Zobnin).
Let $f \in \mathcal{F}\{y\}$ be a first-order differential polynomial not in $\mathcal{F}$.
The ideal $[f]$ has a finite DSB w.r.t. $\prec$ (i.e., $[f]$ contains
a $\prec$-quasi-linear polynomial) iff $[f]$ is radical.

Example.
Let $f_{m,n} = (y_1 + 1)^m - c\,y^n, \quad c \in \mathcal{F},\ c \neq 0.$
Then $[f_{m,n}]$ is radical and has a finite lex-DSB iff $n \mid m$.

Other orderings: a conjecture

Conjecture (M. V. Kondratieva, A. Zobnin). A proper ideal I has a finite DSB w.r.t. a concordant with quasi-linearity β-ordering $\prec$ iff either

- I contains a $\prec$-quasi-linear polynomial, or
- $I = [f^p]$, where f is $\prec$-quasi-linear and $p \geq 1$.

The sufficiency ($\Longrightarrow$) is easy to prove

The necessity ($\Longleftarrow$) is still an open problem.

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