Differential Algebraic Subgroups of $SL(2)$

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Let $F$ be a (partial differential) field of characteristic zero, with a set $\Delta = \{ \delta_1, \ldots, \delta_m \}$ of derivations and let $\mathcal{C}$ be the field of constants of $F$. 

Let $U$ be a universal differential extension of $F$ and let $K$ be the field of constants of $U$. 

$K$ and $F$ are linearly disjoint over $\mathcal{C}$. 

Let $G$ be an extension of $F$, over which $U$ is still universal and let $\mathcal{D}$ be its field of constants. 

An isomorphism of $G$ is a differential field isomorphism from $G$ onto its image $G^\sigma$ in $U$. 

The compositum of two $\Delta$-subfields of $U$ is denoted by concatenation.
We say an isomorphism $\sigma$ of $G$ specializes to another isomorphism $\sigma'$ of $G$ if for every finite family $(\alpha_1, \ldots, \alpha_k) \in G^k$, $(\sigma \alpha_1, \ldots, \sigma \alpha_k)$ specializes (differentially) over $G$ to $(\sigma' \alpha_1, \ldots, \sigma' \alpha_k)$.

Specialization $\sigma \rightarrow \sigma'$ is a reflexive and transitive relation.

Generic specialization: $\sigma \leftrightarrow \sigma'$

$\sigma$ is isolated over $F$ if every specialization from any isomorphism to $\sigma$ is generic.
Let $G$ be a finitely generated extension of $F$, and let $G = F\langle \eta_1, \ldots, \eta_n \rangle$.

$\sigma$ specializes to $\sigma'$ if and only if $(\sigma \eta_1, \ldots, \sigma \eta_n)$ (differentially) specializes to $(\sigma' \eta_1, \ldots, \sigma' \eta_n)$ over $G$.

There exist finitely many isolated isomorphisms $\sigma_1, \ldots, \sigma_k$ of $G$ over $F$ such that every isomorphism of $G$ over $F$ is a specialization of one and only one of these. The number $k$ is unique.

The differential field of invariants of $\sigma_1, \ldots, \sigma_k$ is $F$.

If $F^\circ$ is the algebraic closure of $F$ in $G$, then the differential field of invariants of the component of the identity (the specializations of say $\sigma_1$, where $\sigma_1$ specializes to the identity) is $F^\circ$. 
Let $\sigma$ be an isomorphism of $\mathcal{G}$ over $\mathcal{D}$, and let $\mathcal{D}_\sigma$ be the field of constants of $\mathcal{G}\mathcal{G}^\sigma$. The following conditions are equivalent:

1. $\mathcal{G}^\sigma \subset \mathcal{G}\mathcal{K}$ and $\mathcal{G} \subset \mathcal{G}^\sigma\mathcal{K}$
2. $\mathcal{G}\mathcal{K} = \mathcal{G}^\sigma\mathcal{K}$
3. $\mathcal{G}\mathcal{D}_\sigma = \mathcal{G}\mathcal{G}^\sigma = \mathcal{G}^\sigma\mathcal{D}_\sigma$

An isomorphism $\sigma$ satisfying the conditions is said to be strong (e.g. automorphisms). Let $\text{SI}(\mathcal{G}) =$ set of strong isoms.

$\sigma$ strong implies $\text{tr deg } \mathcal{G}\mathcal{G}^\sigma / \mathcal{G} = \text{tr deg } \mathcal{D}_\sigma / \mathcal{D}$.

$\text{SI}(\mathcal{G}) \leftrightarrow \text{Aut}(\mathcal{G}\mathcal{K} / \mathcal{K})$.

Every specialization of a strong isomorphism is strong.
A strongly normal extension of $\mathcal{F}$ is a $\Delta$-finitely generated extension $\mathcal{G}$ for which every isomorphism of $\mathcal{G}$ over $\mathcal{F}$ is strong.

If $\mathcal{G}$ is finitely generated over $\mathcal{F}$ with $\mathcal{D} = \mathcal{C}$, and if $\mathcal{G}^{\sigma_i} \subset \mathcal{GK}$ for $1 \leq i \leq k$, then $\mathcal{G}$ is strongly normal over $\mathcal{F}$.

Let $\mathcal{G}$ be a strongly normal extension of $\mathcal{F}$.

$\mathcal{D} = \mathcal{C}$.

$\mathcal{G}$ is finitely generated as a field over $\mathcal{F}$, and $\mathcal{D}_\sigma = \mathcal{C}_\sigma$ is finitely generated over $\mathcal{C}$ for every isomorphism $\sigma$ over $\mathcal{F}$.

The set $\text{SI}(\mathcal{G}/\mathcal{F})$ of strong isomorphisms of $\mathcal{G}$ over $\mathcal{F}$, when identified as $\text{Aut}(\mathcal{GK}/\mathcal{FK})$ has the structure of a $\mathcal{C}$-group, now denoted by $G(\mathcal{G}/\mathcal{F})$.

The field associated to a “point” $\sigma$ is $\mathcal{C}_\sigma$. 
(primitive) Let $\mathcal{G} = \mathcal{F} \langle \alpha \rangle$ where $\delta \alpha \in \mathcal{F}$ for all $\delta \in \Delta$. If $\mathcal{D} = \mathcal{C}$, then $\mathcal{G}$ is strongly normal over $\mathcal{F}$. We have $\delta(\sigma \alpha) = \sigma(\delta \alpha) = \delta \alpha$ and hence $c(\sigma) = \sigma \alpha - \alpha \in \mathcal{K}$ and $\sigma$ is strong, with $\mathcal{C}_\sigma = \mathcal{C}(c(\sigma))$ and $c : G(\mathcal{G}/\mathcal{F}) \rightarrow \mathcal{K}$ is an injective $\mathcal{C}$-homomorphism.

(exponential) Let $\mathcal{G} = \mathcal{F} \langle \alpha \rangle$ where $\ell \Delta(\alpha) \in \mathcal{F}^m$. If $\mathcal{D} = \mathcal{C}$, then $\mathcal{G}$ is a strongly normal extension of $\mathcal{F}$. We have $$(\sigma \alpha)^{-1} \delta(\sigma \alpha) = \sigma(\alpha^{-1} \delta \alpha) = \alpha^{-1} \delta \alpha$$ and hence $c(\sigma) = \alpha^{-1} \sigma \alpha \in \mathcal{K}^*$ and $\sigma$ is strong, with $\mathcal{C}_\sigma = \mathcal{C}(c(\sigma))$ and $c : G(\mathcal{G}/\mathcal{F}) \rightarrow \mathcal{K}^*$ is an injective $\mathcal{C}$-homomorphism.
Let $\alpha \in GL(n)$ be a matrix such that $\ell \Delta \alpha = (\Delta \alpha)\alpha^{-1}$ has entries in $\mathcal{F}$. Then $\mathcal{G} = \mathcal{F}\langle \alpha \rangle$ is strongly normal over $\mathcal{F}$. The matrix $c(\sigma) = \alpha^{-1}\sigma\alpha \in GL_K(n)$ and $\sigma$ is strong, with $C_\sigma = C(c(\sigma))$ and $c : G(\mathcal{G}/\mathcal{F}) \to GL_K(n)$ is an injective $\mathcal{C}$-homomorphism.

Let $g_2, g_3 \in \mathcal{C}$ be such that $g_2^3 - 27g_3^2 \neq 0$. Let $\mathcal{G} = \mathcal{F}\langle \alpha \rangle$, where $\alpha$ satisfies, for some $a_1, \ldots, a_m \in \mathcal{F}$, the system:

\[(\delta_i \alpha)^2 = a_i^2(4\alpha^3 - g_2\alpha - g_3), \quad 1 \leq i \leq m.\]

Then $\mathcal{G}$ is strongly normal over $\mathcal{F}$.

Both primitive and exponential extensions are Picard-Vessiot, but if $\alpha$ is transcendental over $\mathcal{F}$, and Weierstraussian, then $\mathcal{G}$ is not a Picard-Vessiot extension of $\mathcal{F}$. 
Let $t \in \mathcal{U}$ be differentially transcendental over $\mathbb{F}$. Let $\mathcal{G} = \mathbb{F}(t)$.

Problem: Find all intermediate differential subfields $\mathcal{E}$ between $\mathbb{F}$ and $\mathcal{G}$ such that $\mathcal{G}$ is strongly normal over $\mathcal{E}$.

Quick answer: $\mathcal{E}$ is generated basically over $\mathbb{F}$ by:

1. Schwarzian derivatives of $t$, $S\delta(t) = \frac{\delta^3 t}{\delta t} - \frac{3(\delta^2 t)^2}{2(\delta t)^2}$

2. logarithmic derivatives of linear homogeneous differential polynomials in $t$ with coefficients in $\mathbb{F}$

3. powers of linear homogeneous differential polynomials in $t$ with coefficients in $\mathbb{F}$

4. polyhedral functions of $t$

5. exceptions involving certain non-zero constants
Let $\mathcal{H}$ be a differential subfield of $\mathcal{U}$ over which $\mathcal{U}$ need not be universal but there is an $s \in \mathcal{U}$ differentially transcendental over $\mathcal{H}$. For example, $\mathcal{H} = \mathcal{F}\mathcal{K}$.

Let $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(n)$. Say $x \in P(\mathcal{H})$ if there exists $u \in \mathcal{U}^*$ such that $ux \in GL_{\mathcal{H}}(n)$.

For any $p \in \mathcal{U}$ such that $-cp + a \neq 0$, define $p^x = \frac{dp - b}{-cp + a} \in \mathcal{U}$.

Define $\tau_x$ to be the automorphism in $\mathcal{H}\langle s \rangle / \mathcal{H}$ with $\tau_x(s) = s^x$. Then $\tau_{xx'} = \tau_x \circ \tau_{x'}$ and if we let $Z = \{ +1, -1 \} \subset SL(2)$, we have an exact sequence of groups:

$$1 \rightarrow Z \rightarrow P(\mathcal{H}) \xrightarrow{\tau} \text{Aut}(\mathcal{H}\langle s \rangle / \mathcal{H}) \rightarrow 1.$$
A differential algebraic group is a differentially closed subset $G$ of $U^n$ such that $G$ is a group and the group laws are everywhere defined differential rational maps.

$G$ is defined over $F$ if the differentially closed set $G$ and the group laws are defined over $F$ (that is, given by differentially polynomial or rational functions with coefficients in $F$).

(Cassidy) The components of $G$ have the same differential dimension, are mutually disjoint, and are cosets of the component $G^\circ$ containing the identity, which is a connected, normal differential algebraic subgroup of $G$.

$G$ is linear if $G \subset GL(n)$. Viewed as a differential algebraic subgroup of $SL(n + 1)$, the group laws are polynomials maps.

(Cassidy) Every differential algebraic group whose group laws are given by differential polynomial functions is isomorphic to a linear differential algebraic group.
Galois Groups and $SL(2)$

- $G(\mathcal{G}/\mathcal{E})$ is identified with a subgroup of $\text{Aut}(\mathcal{G}\mathcal{K}/\mathcal{F}\mathcal{K}) = \text{Aut}(\mathcal{F}\mathcal{K}\langle t \rangle/\mathcal{F}\mathcal{K})$.

- Any $\sigma \in \text{Aut}(\mathcal{F}\mathcal{K}\langle t \rangle/\mathcal{F}\mathcal{K})$ can be represented by a matrix in $P(\mathcal{F}\mathcal{K}) \subset SL(2)$, or a matrix in $GL_{\mathcal{F}\mathcal{K}}(2)$.

- There is a unique differential algebraic subgroup $H(\mathcal{E})$ of $SL(2)$, $\mathcal{Z} \subset H(\mathcal{E}) \subset P(\mathcal{F}\mathcal{K})$, such that the sequence is exact:

$$1 \to \mathcal{Z} \to H(\mathcal{E}) \xrightarrow{\tau} G(\mathcal{G}/\mathcal{E}) \to 1.$$  

Moreover, the fixed field of $H(\mathcal{E})$ is $\mathcal{E}$.

- (Sketch) $H(\mathcal{E})$ is the set of $x \in SL(2)$ satisfying the system obtained after clearing denominators of the following:

$$A_{\eta}(y^x)B_{\eta}(y) = A_{\eta}(y)B_{\eta}((y^x)), \quad \eta \in \mathcal{E}$$

where $\eta = A_{\eta}(t)/B_{\eta}(t)$ and $y$ is a differential indeterminate over $\mathcal{U}$. This just says $\sigma_x$ fixes every $\eta \in \mathcal{E}$ when $y \mapsto t$.  

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Goal: Compute all subfields $\mathcal{E}$ such that $\mathcal{G}$ is strongly normal over $\mathcal{E}$.

(Strategy):

1. Compute all differential algebraic subgroups $H$ of $SL(2)$.
2. Identify those satisfying $\mathbb{Z} \subset H \subset P(\mathbb{K})$.
3. For each such subgroup $H$, compute its fixed field $\mathcal{E}(H)$.
4. Prove that $\mathcal{G}$ is strongly normal over each $\mathcal{E}(H)$.
5. Compute its Galois group $G(\mathcal{G}/\mathcal{E}(H))$.
6. Show that its Galois group is $H$.

Suffices to do this up to conjugation over $\mathbb{F}$, which means we need to first classify all differential algebraic subgroups of $SL(2)$.

The Zariski closure $G$ of a differential algebraic subgroup $H$ defined over $\mathbb{F}$ is an algebraic subgroup, also defined over $\mathbb{F}$.
Infinite Algebraic Subgroups of $SL(2)$

- Up to conjugation, the infinite algebraic subgroups of $SL(2)$ defined over a field $K$ are classified into five types.

1. $\dim G = 3$; then $G = SL(2)$.
2. $\dim G = 2$; then $G$ is connected and conjugate over $K$ to $ST(2)$, the special triangular subgroup.
3. $\dim G = 1$ and $G^\circ$ is unipotent; then $G$ is conjugate over $K$ to $U^n(2)$ for some positive integer $n$.
4. $\dim G = 1$, $G$ is connected and diagonalizable; then $G$ is conjugate over $K$ to $SO^e(2)$ for some $e \in K^*$.
5. $\dim G = 1$, $G \neq G^\circ$, and $G^\circ$ is diagonalizable; then $G$ is conjugate over $K$ to $SO^e(2)^\dagger$ for some $e \in K^*$.

- Here, $H$ and $H'$, subsets of $SL(2)$, are conjugate over $K$ if there exists $z \in P(K)$ such that $zHz^{-1} = H'$.

- \( U^n(2) \) \:
  \[ \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}, \alpha^n = 1 \right\} \]

- \( SO^e(2), SO^e(2)^\dagger \) \:
  \[ \left\{ \begin{pmatrix} \alpha & \pm e\gamma \\ \pm e\alpha & \alpha \end{pmatrix}, \alpha^2 - e\gamma^2 = \pm 1 \right\} . \]
Let $G$ be a finite subgroup of $SL(2)$ or order $n$. Then there are five possibilities:

1. $G$ is cyclic.
2. $G/Z \subset G$ and $G/Z$ is dihedral.
3. $G/Z \subset G$ and $G/Z$ is isomorphic to $A_4$; tetrahedral.
4. $G/Z \subset G$ and $G/Z$ is isomorphic to $S_4$; octahedral.
5. $G/Z \subset G$ and $G/Z$ is isomorphic to $A_5$; icosahedral.

If $K$ contains appropriate roots of unity, and if $G \subset P(K)$, then $G$ is defined over $K$ and is conjugate over $K$ to one of the groups: $C(n)$, $D(4m, e)$, $D(8, e, f, g)$, $T$, $OC$, $DI$. 
Let \( D = U \Delta \), the vector space over \( U \) with basis \( \Delta \). Let \( \mathcal{E} \) be a subspace of dimension \( k \), and let \( \mathcal{K}(\mathcal{E}) = \bigcap_{E \in \mathcal{E}} \text{Ker} \ E \) be its field of constants, which is clearly algebraically closed.

1. \( \mathcal{E} \) is a Lie subspace if the following equivalent conditions hold:
   1. for all \( \mathcal{D} \in \mathcal{D} \) and \( \mathcal{D} \in \mathcal{E} \) implies \( \mathcal{D} \in \mathcal{E} \).
   2. \( \mathcal{E} \) has a commuting basis \( \mathcal{E}_1, \ldots, \mathcal{E}_k \), where \( [E_i, E_j] = 0 \) for all \( i, j \).

2. \( \mathcal{E} = \sum_{i=1}^{k} a_i \mathcal{E}_i \in \mathcal{E} \) is rational over \( F \) relative to a basis \( \mathcal{E}_1, \ldots, \mathcal{E}_k \).

3. \( \mathcal{E} \) is defined over \( F \) if it has a basis where \( \mathcal{E}_i \) is rational over \( F \) relative to \( \Delta \).

4. If \( \mathcal{E} \) is defined over \( F \), then rationality of \( \mathcal{E} \in \mathcal{E} \) relative to a rational basis is rationality relative to \( \Delta \).

The subset of \( \mathcal{E} \) consisting of all rational operators over \( F \) is denoted by \( \mathcal{E}_F \).
Let $\mathcal{D} = \mathcal{U}\Delta$, the vector space over $\mathcal{U}$ with basis $\Delta$. Let $\mathcal{E}$ be a subspace of dimension $k$, and let $\mathcal{K}(\mathcal{E}) = \cap_{E \in \mathcal{E}} \text{Ker } E$ be its field of constants, which is clearly algebraically closed.

$\mathcal{E}$ is Lie subspace if the following equivalent conditions hold:

1. $D \in \mathcal{D}$ and $D(\mathcal{K}(\mathcal{E})) = 0$ implies $D \in \mathcal{E}$.
2. $\mathcal{E}$ has a commuting basis $E_1, \ldots, E_k$, where $[E_i, E_j] = 0$ for all $i, j$. 

$\mathcal{E}$ is defined over $\mathcal{F}$ if it has a basis where $E_i$ is rational over $\mathcal{F}$ relative to $\Delta$.

If $\mathcal{E}$ is defined over $\mathcal{F}$, then rationality of $E \in \mathcal{E}$ relative to a rational basis is rationality relative to $\Delta$. The subset of $\mathcal{E}$ consisting of all rational operators over $\mathcal{F}$ is denoted by $\mathcal{E}_F$. 

\[ \text{Lie Vector Spaces and Fields of Constants} \]
Lie Vector Spaces and Fields of Constants

Let $\mathcal{D} = \mathcal{U}\Delta$, the vector space over $\mathcal{U}$ with basis $\Delta$. Let $\mathcal{E}$ be a subspace of dimension $k$, and let $\mathcal{K}(\mathcal{E}) = \bigcap_{E \in \mathcal{E}} \text{Ker } E$ be its field of constants, which is clearly algebraically closed.

$\mathcal{E}$ is Lie subspace if the following equivalent conditions hold:

1. $D \in \mathcal{D}$ and $D(\mathcal{K}(\mathcal{E})) = 0$ implies $D \in \mathcal{E}$.
2. $\mathcal{E}$ has a commuting basis $E_1, \ldots, E_k$, where $[E_i, E_j] = 0$ for all $i, j$.

$E = \sum_{i=1}^{k} a_i E_i \in \mathcal{E}$ is rational over $\mathcal{F}$ relative to a basis $E_1, \ldots, E_k$ if $a_i \in \mathcal{F}$ for all $i$. $\mathcal{E}$ is defined over $\mathcal{F}$ if it has a basis where $E_i$ is rational over $\mathcal{F}$ relative to $\Delta$. 
Lie Vector Spaces and Fields of Constants

Let \( D = U \Delta \), the vector space over \( U \) with basis \( \Delta \). Let \( E \) be a subspace of dimension \( k \), and let \( \mathcal{K}(E) = \cap_{E \in E} \text{Ker} \ E \) be its field of constants, which is clearly algebraically closed.

\( E \) is Lie subspace if the following equivalent conditions hold:

1. \( D \in D \) and \( D(\mathcal{K}(E)) = 0 \) implies \( D \in E \).
2. \( E \) has a commuting basis \( E_1, \ldots, E_k \), where \( [E_i, E_j] = 0 \) for all \( i, j \).

\( E = \sum_{i=1}^{k} a_i E_i \in E \) is rational over \( F \) relative to a basis \( E_1, \ldots, E_k \) if \( a_i \in F \) for all \( i \). \( E \) is defined over \( F \) if it has a basis where \( E_i \) is rational over \( F \) relative to \( \Delta \).

If \( E \) is defined over \( F \), then rationality of \( E \in E \) relative to a rational basis is rationality relative to \( \Delta \). The subset of \( E \) consisting of all rational operators over \( F \) is denoted by \( E_F \).
Let $\mathcal{E}$ be Lie subspace of $\mathcal{D}$, defined over $\mathcal{F}$ and let $s \in SL(n)$ be such that $\ell E(s) = (E s)s^{-1} \in sl_{\mathcal{F}}(n)$ for all $E \in \mathcal{E}_{\mathcal{F}}$. Let $SL(\mathcal{E}, s) = s SL_{K(\mathcal{E})}(n)s^{-1}$. 
Let $\mathcal{E}$ be Lie subspace of $\mathcal{D}$, defined over $\mathcal{F}$ and let $s \in SL(n)$ be such that $\ell E(s) = (Es)s^{-1} \in sl_{\mathcal{F}}(n)$ for all $E \in \mathcal{E}_{\mathcal{F}}$. Let $SL(\mathcal{E}, s) = sSL_{K(\mathcal{E})}(n)s^{-1}$.

(Cassidy) $SL(\mathcal{E}, s)$ is a Zariski dense differential algebraic subgroup of $SL(n)$ and is defined over $\mathcal{F}$ and these are all.
Let $\mathcal{E}$ be Lie subspace of $\mathcal{D}$, defined over $\mathcal{F}$ and let $s \in SL(n)$ be such that $\ell E(s) = (Es)s^{-1} \in sl_{\mathcal{F}}(n)$ for all $E \in \mathcal{E}_{\mathcal{F}}$. Let $SL(\mathcal{E}, s) = sSL_{\mathcal{K}(\mathcal{E})}(n)s^{-1}$.

(Cassidy) $SL(\mathcal{E}, s)$ is a Zariski dense differential algebraic subgroup of $SL(n)$ and is defined over $\mathcal{F}$ and these are all.

$SL(\mathcal{E}, s)$ is conjugate over $\mathcal{F}$ to $SL(\mathcal{E}', s')$ if and only if $\mathcal{E} = \mathcal{E}'$ and there exists $z \in P(\mathcal{F})$ such that $s'^{-1}zs \in SL_{\mathcal{K}(\mathcal{E})}(n)$. 

Example. Let $\mathcal{F} = \mathbb{Q}(i \sin x, i \cos x)$, $\delta = \frac{d}{dx}$, $i = \sqrt{-1}$. Then $C = \mathbb{Q}$. Fix elements $\alpha, \gamma \in U$ such that $\alpha^2 - \gamma^2 = \sin x$, $2\alpha\gamma = -\cos x$, $\alpha^2 + \gamma^2 = -1$. Let $s = \left( \begin{array}{cc} \alpha & \gamma \\ \gamma & -\alpha \end{array} \right)$. Then $\ell_\delta s = \left( \begin{array}{ccc} 0 & -1/2 & 1/2 \\ 1/2 & 0 & -1/2 \\ 0 & 1/2 & 0 \end{array} \right)$ and $H = SL(D, s)$ is a Zariski dense differential algebraic subgroup, contained in $P(\mathcal{F})$, but not conjugate over $\mathcal{F}$ to $SL_{\mathcal{K}(\mathcal{E})}(n)$. 

(Cassidy) Zariski Dense Subgroups of $SL(n)$
Let $\mathcal{E}$ be a Lie subspace of $\mathcal{D}$, defined over $\mathcal{F}$ and let $s \in SL(n)$ be such that $\ell E(s) = (Es)s^{-1} \in sl_\mathcal{F}(n)$ for all $E \in \mathcal{E}_\mathcal{F}$. Let $SL(\mathcal{E}, s) = sSL_K(\mathcal{E})(n)s^{-1}$.

$(\text{Cassidy})$ $SL(\mathcal{E}, s)$ is a Zariski dense differential algebraic subgroup of $SL(n)$ and is defined over $\mathcal{F}$ and these are all.

$SL(\mathcal{E}, s)$ is conjugate over $\mathcal{F}$ to $SL(\mathcal{E}', s')$ if and only if $\mathcal{E} = \mathcal{E}'$ and there exists $z \in \mathcal{P}(\mathcal{F})$ such that $s'^{-1}zs \in SL_K(\mathcal{E})(n)$.

Example. Let $\mathcal{F} = \mathbb{Q}(i \sin x, i \cos x)$, $\delta = d/dx$, $i = \sqrt{-1}$. Then $\mathcal{E} = \mathbb{Q}$. Fix elements $\alpha, \gamma \in \mathfrak{U}$ such that

$$\alpha^2 - \gamma^2 = \sin x, \quad 2\alpha \gamma = -\cos x, \quad \alpha^2 + \gamma^2 = -1.$$ 

Let $s = \begin{pmatrix} \alpha & \gamma \\ \gamma & -\alpha \end{pmatrix}$. Then $\ell \delta s = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$ and $H = SL(\mathcal{D}, s)$ is a Zariski dense differential algebraic subgroup, contained in $\mathcal{P}(\mathcal{F}K)$, but not conjugate over $\mathcal{F}$ to $SL_K(2)$. 
[Cassidy] Every linear differential ideal \( \mathfrak{p} \) of \( \mathcal{U}\{y_1, \ldots, y_n\} \) defines a differential algebraic subgroup \( B \) of \( \mathbb{G}_a^n \), which is connected, is a vector subspace of \( \mathcal{U}^n \) over \( \mathcal{K} \), and is defined over \( \mathcal{F} \) if and only if \( \mathfrak{p} \) is. Moreover, these are all.
Every linear differential ideal $\mathfrak{p}$ of $\mathcal{U}\{y_1, \ldots, y_n\}$ defines a differential algebraic subgroup $B$ of $\mathbb{G}_a^n$, which is connected, is a vector subspace of $\mathcal{U}^n$ over $\mathcal{K}$, and is defined over $\mathcal{F}$ if and only if $\mathfrak{p}$ is. Moreover, these are all.

The set $\mathcal{K}(B)$ of all $a \in \mathcal{U}$ such that $aB \subseteq B$ is the field of constants $\mathcal{K}(\mathcal{E})$ of a unique Lie subspace $\mathcal{E}$ of $\mathcal{D}$, which is defined over $\mathcal{F}$. Note that $\mathcal{K}(0) = \mathcal{U}$ when $B = 0$ or $\mathcal{E} = 0$. 

$\mathbb{G}^n_{a}$
(Cassidy) Every linear differential ideal \( p \) of \( \mathcal{U}\{y_1, \ldots, y_n\} \) defines a differential algebraic subgroup \( B \) of \( \mathbb{G}_{\mathbb{A}}^n \), which is connected, is a vector subspace of \( \mathcal{U}^n \) over \( \mathcal{K} \), and is defined over \( \mathcal{F} \) if and only if \( p \) is. Moreover, these are all.

The set \( \mathcal{K}(B) \) of all \( a \in \mathcal{U} \) such that \( aB \subseteq B \) is the field of constants \( \mathcal{K}(\mathcal{E}) \) of a unique Lie subspace \( \mathcal{E} \) of \( \mathcal{D} \), which is defined over \( \mathcal{F} \). Note that \( \mathcal{K}(0) = \mathcal{U} \) when \( B = 0 \) or \( \mathcal{E} = 0 \).

Relative to any orderly ranking, \( p \) has a canonical, linear, homogeneous characteristic set \( B \subseteq \mathcal{F}\{y_1, \ldots, y_n\} \) such that

\[ L(ay_1, \ldots, ay_n) = aL(y_1, \ldots, y_n), \quad (L \in B, a \in \mathcal{K}(B)). \]

Indeed, for every \( L \in p \cap \mathcal{F}\{y_1, \ldots, y_n\} \).
(Cassidy) Every linear differential ideal \( p \) of \( \mathcal{U}\{y_1, \ldots, y_n\} \) defines a differential algebraic subgroup \( B \) of \( \mathbb{G}^n_a \), which is connected, is a vector subspace of \( \mathcal{U}^n \) over \( \mathcal{K} \), and is defined over \( \mathcal{F} \) if and only if \( p \) is. Moreover, these are all.

The set \( \mathcal{K}(B) \) of all \( a \in \mathcal{U} \) such that \( aB \subseteq B \) is the field of constants \( \mathcal{K}(\mathcal{E}) \) of a unique Lie subspace \( \mathcal{E} \) of \( \mathcal{D} \), which is defined over \( \mathcal{F} \). Note that \( \mathcal{K}(0) = \mathcal{U} \) when \( B = 0 \) or \( \mathcal{E} = 0 \).

Relative to any orderly ranking, \( p \) has a canonical, linear, homogeneous characteristic set \( B \subset \mathcal{F}\{y_1, \ldots, y_n\} \) such that

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L(ay_1, \ldots, ay_n) = aL(y_1, \ldots, y_n), \quad (L \in B, a \in \mathcal{K}(B)).
\]

Indeed, for every \( L \in p \cap \mathcal{F}\{y_1, \ldots, y_n\} \).

An element \( \beta \in \mathcal{U}^n \) is said to be rational over \( \mathcal{F} \) mod \( B \), that is, the coset \( \beta + B \in \mathbb{G}^n_a/B \) is rational over \( \mathcal{F} \), if \( L(\beta) \in \mathcal{F} \) for every linear homogeneous differential polynomial \( L \in p \); or equivalently, if \( L(\beta) \in \mathcal{F} \) for every \( L \in B \). If \( B = 0 \), \( \beta \in \mathcal{F} \).
The logarithmic derivative map $\ell \Delta : G_m \rightarrow G_a^m$ given by $\ell \Delta x = (\frac{\delta_1 x}{x}, \ldots, \frac{\delta_m x}{x})$ is a differential rational homomorphism defined over $\mathbb{Q}$ with kernel $G_m\mathcal{K} = \mathcal{K}^*$ and image $I$, which is a differential algebraic subgroup of $G_a^m$ with corresponding linear differential ideal generated by $\{ \delta_i y_j - \delta_j y_i \mid 1 \leq i < j \leq m \}$. 
The logarithmic derivative map $\ell \Delta : \mathbb{G}_m \rightarrow \mathbb{G}_a^m$ given by $\ell \Delta x = (\frac{\delta_1 x}{x}, \ldots, \frac{\delta_m x}{x})$ is a differential rational homomorphism defined over $\mathbb{Q}$ with kernel $\mathbb{G}_m K = K^*$ and image $I$, which is a differential algebraic subgroup of $\mathbb{G}_a^m$ with corresponding linear differential ideal generated by $\{ \delta_i y_j - \delta_j y_i \mid 1 \leq i < j \leq m \}$.

(Cassidy) Every infinite differential algebraic subgroup $A$ of $\mathbb{G}_m$ contains $\mathbb{G}_m K$, is connected, corresponds to a differential algebraic subgroup of $I$, and is defined over $\mathcal{F}$ if and only if $\ell \Delta(A)$ is. These are all.
The logarithmic derivative map $\ell \Delta : \mathbb{G}_m \to \mathbb{G}_a^m$ given by $\ell \Delta x = (\frac{\delta_1 x}{x}, \ldots, \frac{\delta_m x}{x})$ is a differential rational homomorphism defined over $\mathbb{Q}$ with kernel $\mathbb{G}_m \mathbb{K} = \mathbb{K}^*$ and image $\mathbb{I}$, which is a differential algebraic subgroup of $\mathbb{G}_a^m$ with corresponding linear differential ideal generated by $\{ \delta_i y_j - \delta_j y_i \mid 1 \leq i < j \leq m \}$.

(Cassidy) Every infinite differential algebraic subgroup $A$ of $\mathbb{G}_m$ contains $\mathbb{G}_m \mathbb{K}$, is connected, corresponds to a differential algebraic subgroup of $\mathbb{I}$, and is defined over $\mathcal{F}$ if and only if $\ell \Delta (A)$ is. These are all.

The finite (differential) algebraic subgroups of $\mathbb{G}_m$ are the cyclic groups $\mathbb{P}_n$ consisting of $n$-th roots of unity.
B is a differential algebraic subgroup of $G_a$ defined over $\mathcal{F}$; $A$ is a (finite or infinite) differential algebraic subgroup of $G_{m_k(B)} = k(B)^*$, also defined over $\mathcal{F}$; $\beta \in \mathcal{U}$ is rational over $\mathcal{F}$ mod $B$. 
B is a differential algebraic subgroup of $G_a$ defined over $F$; 
$A$ is a (finite or infinite) differential algebraic subgroup of 
$G_{m,K(B)} = K(B)^*$, also defined over $F$; 
$\beta \in U$ is rational over $F$ mod $B$.

To such a triple $(B, A, \beta)$, we define

$$ST(B, A, \beta) = \left\{ \begin{pmatrix} a & (a - a^{-1})\beta + b \\ 0 & a^{-1} \end{pmatrix} \bigg| a \in A, b \in B \right\}.$$
Differential Algebraic Subgroups of $ST(2)$

♦ $B$ is a differential algebraic subgroup of $\mathbb{G}_a$ defined over $\mathcal{F}$; 
$A$ is a (finite or infinite) differential algebraic subgroup of $\mathbb{G}_{m,\mathcal{K}(B)} = \mathcal{K}(B)^*$, also defined over $\mathcal{F}$; 
$\beta \in \mathcal{U}$ is rational over $\mathcal{F}$ mod $B$.

♦ To such a triple $(B, A, \beta)$, we define

$$ST(B, A, \beta) = \left\{ \begin{pmatrix} a & (a - a^{-1})\beta + b \\ 0 & a^{-1} \end{pmatrix} \middle| a \in A, b \in B \right\}.$$ 

♦ If $A = 1$ or $A = \mathbb{P}_2$, we only define this when $\beta = 0$. 
Differential Algebraic Subgroups of $ST(2)$

- $B$ is a differential algebraic subgroup of $G_a$ defined over $\mathcal{F}$; $A$ is a (finite or infinite) differential algebraic subgroup of $G_{m, \mathcal{K}(B)} = \mathcal{K}(B)^*$, also defined over $\mathcal{F}$; $\beta \in \mathcal{U}$ is rational over $\mathcal{F}$ mod $B$.

- To such a triple $(B, A, \beta)$, we define
  
  $$ST(B, A, \beta) = \left\{ \begin{pmatrix} a & (a - a^{-1})\beta + b \\ 0 & a^{-1} \end{pmatrix} \bigg| a \in A, b \in B \right\}.$$

- If $A = 1$ or $A = \mathbb{P}_2$, we only define this when $\beta = 0$.

- $ST(B, A, \beta)$ is a differential algebraic subgroup of $ST(2)$ defined over $\mathcal{F}$, and these are all. Moreover,
  
  $$ST(B, A, \beta) = ST(B, 1, 0) \cdot ST(0, A, \beta) \quad \text{(semidirect product)}$$
Differential Algebraic Subgroups of $ST(2)$

- $B$ is a differential algebraic subgroup of $G_a$ defined over $F$; $A$ is a (finite or infinite) differential algebraic subgroup of $G_{m,K(B)} = K(B)^*$, also defined over $F$; and $\beta \in U$ is rational over $F \mod B$.

- To such a triple $(B, A, \beta)$, we define

$$ST(B, A, \beta) = \left\{ \begin{pmatrix} a & (a-a^{-1})\beta + b \\ 0 & a^{-1} \end{pmatrix} \middle| a \in A, b \in B \right\}.$$

- If $A = 1$ or $A = P_2$, we only define this when $\beta = 0$.

- $ST(B, A, \beta)$ is a differential algebraic subgroup of $ST(2)$ defined over $F$, and these are all. Moreover,

$$ST(B, A, \beta) = ST(B, 1, 0) \cdot ST(0, A, \beta) \quad \text{(semidirect product)}.$$

- $ST(B, A, \beta)$ is conjugate over $F$ to $ST(B', A', \beta')$ if and only if $A = A'$, and there exists $f \in F^*$ such that $fB = B'$, and $(f\beta - \beta' + B') \cap F \neq \emptyset$. 
For all $H = ST(B, A, \beta)$, one and only one of the following holds:

1. $B \neq 0$ and $A$ is infinite; dense in $ST(2)$.
2. $B \neq 0$ and $A = P_n$; dense in $U^n(2)$.
3. $B = 0$ and $A$ is infinite; then $\beta \in F$, $H$ is conjugate over $F$ to $ST(0, A, 0)$, which is dense in $SD(2)$.
4. $B = 0$ and $A = P_n$; then $\beta \in F$, $H$ is finite and conjugate over $F$ to $C(n)$. 
Recall for $e \in \mathcal{F}^*$, $SO^e(2) = \left\{ \begin{pmatrix} \alpha & e\gamma \\ \gamma & \alpha \end{pmatrix} \bigg| \alpha^2 - e\gamma^2 = 1 \right\}$. 
Recall for \( e \in \mathcal{F}^* \), \( SO^e(2) = \left\{ \begin{pmatrix} \alpha & e\gamma \\ \gamma & \alpha \end{pmatrix} \bigg| \alpha^2 - e\gamma^2 = 1 \right\} \).

Let \( e^2 = e \), \( s = \begin{pmatrix} 1 & e \\ -1 & e \end{pmatrix} \) and let \( \tau_s(u) = sus^{-1}, \ell_e(x) = \epsilon x \).

Then the map \( \pi_e : SO^e(2) \xrightarrow{\tau_s} SD(2) \xrightarrow{p} G_m \xrightarrow{\ell_\Delta} G^m_a \xrightarrow{\ell_e} G^m_a \) is a differential rational homomorphism defined over \( \mathbb{Q} \), given by

\[
\pi_e\left( \begin{pmatrix} \alpha & e\gamma \\ \gamma & \alpha \end{pmatrix} \right) = \begin{cases} 
\left( \frac{\delta_1\alpha}{\gamma}, \ldots, \frac{\delta_m\alpha}{\gamma} \right) & \text{if } \gamma \neq 0; \\
(0, \ldots, 0) & \text{if } \gamma = 0.
\end{cases}
\]

with

\[
\ker \pi_e = \left\{ \begin{pmatrix} \alpha & e\gamma \\ \gamma & \alpha \end{pmatrix} \bigg| \alpha \in \mathcal{K}, \gamma \in \mathcal{U}, \alpha^2 - e\gamma^2 = 1 \right\}.
\]

Image = differential algebraic \( \mathcal{F} \)-subgroup \( l_e \) of \( G^m_a \) defined by

\[
\delta_i y_j - \delta_j y_i + \frac{1}{2}(\ell \delta_j e)y_i - \frac{1}{2}(\ell \delta_i e)y_j \quad (1 \leq i < j \leq m).
\]
Dense Differential Algebraic Subgroups of $\text{SO}^e(2)$

$p \circ \tau_s : \text{SO}^e(2) \rightarrow \mathbf{G}_m$ is a rational isomorphism;

$\begin{pmatrix} \alpha & e\gamma \\ \gamma & \alpha \end{pmatrix} \mapsto \alpha + \epsilon \gamma.$
Dense Differential Algebraic Subgroups of $SO^e(2)$

$p \circ \tau_s : SO^e(2) \to G_m$ is a rational isomorphism;
\[
\begin{pmatrix}
\alpha & e\gamma \\
\gamma & \alpha
\end{pmatrix} \mapsto \alpha + \epsilon \gamma.
\]

- Exact sequence: $1 \to \text{Ker} \pi_e \to SO^e(2) \xrightarrow{\pi_e} \text{I}_e \to 1$.

- There is a bijection between the set of differential algebraic subgroups $B$ of $\text{I}_e$ and the set of differential algebraic subgroup of $SO^e(2)$ containing $\text{Ker} \pi_e$. This bijection is given by $B \mapsto SO^e(B) = \pi_e^{-1}(B)$, $B$ is defined over $\mathcal{F}$ if and only if $SO^e(B)$ is.
Dense Differential Algebraic Subgroups of $SO^e(2)$

- $p \circ \tau_s : SO^e(2) \to G_m$ is a rational isomorphism; 
  \[
  \begin{pmatrix} \alpha & e\gamma \\ \gamma & \alpha \end{pmatrix} \mapsto \alpha + e\gamma.
  \]

- Exact sequence: $1 \to \text{Ker } \pi_e \to SO^e(2) \xrightarrow{\pi_e} I_e \to 1$.

- There is a bijection between the set of differential algebraic subgroups $B$ of $I_e$ and the set of differential algebraic subgroup of $SO^e(2)$ containing $\text{Ker } \pi_e$. This bijection is given by $B \mapsto SO^e(B) = \pi_e^{-1}(B)$, $B$ is defined over $\mathcal{F}$ if and only if $SO^e(B)$ is.

- A differential algebraic subgroup $H$ is dense in $SO^e(2)$ if and only if $H \supset \text{Ker } \pi_e$, in which case $H = SO^e(B)$ for a unique differential algebraic subgroup $B$ of $I_e$, $H$ is connected, and $H$ is defined over $\mathcal{F}$ if and only if $B$ is. These are all.
Dense Differential Algebraic Subgroups of $SO^e(2)$

• $p \circ \tau_s : SO^e(2) \to \mathbb{G}_m$ is a rational isomorphism;
  \[
  \begin{pmatrix}
  \alpha & e\gamma \\
  \gamma & \alpha
  \end{pmatrix} \mapsto \alpha + e\gamma.
  \]

• Exact sequence: $1 \to \text{Ker} \pi_e \to SO^e(2) \xrightarrow{\pi_e} I_e \to 1$.

• There is a bijection between the set of differential algebraic subgroups $B$ of $I_e$ and the set of differential algebraic subgroup of $SO^e(2)$ containing $\text{Ker} \pi_e$. This bijection is given by $B \mapsto SO^e(B) = \pi_e^{-1}(B)$, $B$ is defined over $\mathcal{F}$ if and only if $SO^e(B)$ is.

• A differential algebraic subgroup $H$ is dense in $SO^e(2)$ if and only if $H \supset \text{Ker} \pi_e$, in which case $H = SO^e(B)$ for a unique differential algebraic subgroup $B$ of $I_e$, $H$ is connected, and $H$ is defined over $\mathcal{F}$ if and only if $B$ is. These are all.

• $SO^e(B)$ and $SO^e(B')$ are conjugate over $\mathcal{F}$ if and only if there exists $f \in \mathcal{F}^*$ such that $e'^{-1}e = f^2$ and $fB' = B$. 
Recall for $e \in \mathcal{F}^*$,

$$SO^e(2)^* = \left\{ \begin{pmatrix} \alpha & -e\gamma \\ \gamma & -\alpha \end{pmatrix} \left| \begin{array}{l} \alpha^2 - e\gamma^2 = -1 \end{array} \right. \right\}.$$
Recall for $e \in \mathcal{F}^*$, 

$$SO^e(2)^* = \left\{ \begin{pmatrix} \alpha & -e\gamma \\ \gamma & -\alpha \end{pmatrix} \bigg| \alpha^2 - e\gamma^2 = -1 \right\}.$$ 

The map $\pi_e^* : SO^e(2)^* \xrightarrow{\tau^*_s} SD(2)^* \xrightarrow{p^*} G_m \xrightarrow{\ell \Delta} G_a^m \xrightarrow{\ell \epsilon} G_a^m$ is a differential rational map defined over $\mathbb{Q}$ given by 

$$\pi_e^*(\begin{pmatrix} \alpha & -e\gamma \\ \gamma & -\alpha \end{pmatrix}) = \begin{cases} \left( \frac{\delta_1 \alpha}{\gamma}, \ldots, \frac{\delta_m \alpha}{\gamma} \right) & \text{if } \gamma \neq 0; \\
(0, \ldots, 0) & \text{if } \gamma = 0. \end{cases}$$

with 

$$\text{Ker } \pi_e^* = \left\{ \begin{pmatrix} \alpha & -e\gamma \\ \gamma & -\alpha \end{pmatrix} \bigg| \alpha \in \mathcal{K}, \gamma \in \mathcal{U}, \alpha^2 - e\gamma^2 = -1 \right\}.$$ 

Image is the differential algebraic subgroup $I_e$ of $G_a^m$. 


Recall for $e \in F^*$,

$$SO^e(2)^* = \left\{ \begin{pmatrix} \alpha & -e\gamma \\ \gamma & -\alpha \end{pmatrix} \bigg| \alpha^2 - e\gamma^2 = -1 \right\}.$$ 

The map $\pi_e^* : SO^e(2)^* \xrightarrow{\tau_s^*} SD(2)^* \xrightarrow{p^*} G_m \xrightarrow{\ell \Delta} G_a^m \xrightarrow{\ell \epsilon} G_a^m$ is a differential rational map defined over $\mathbb{Q}$ given by

$$\pi_e^*\left( \begin{pmatrix} \alpha & -e\gamma \\ \gamma & -\alpha \end{pmatrix} \right) = \begin{cases} \left( \frac{\delta_1 \alpha}{\gamma}, \ldots, \frac{\delta_m \alpha}{\gamma} \right) & \text{if } \gamma \neq 0; \\ (0, \ldots, 0) & \text{if } \gamma = 0. \end{cases}$$

with

$$\text{Ker } \pi_e^* = \left\{ \begin{pmatrix} \alpha & -e\gamma \\ \gamma & -\alpha \end{pmatrix} \bigg| \alpha \in K, \gamma \in U, \alpha^2 - e\gamma^2 = -1 \right\}.$$ 

Image is the differential algebraic subgroup $I_e$ of $G_a^m$.

Note that $p^* \circ \tau_s^*$ is a bijective birational map.
Recall that $SO^e(2)^\dagger = SO^e(2) \cup SO^e(2)^\ast$. 
Recall that $SO^e(2)^\dagger = SO^e(2) \cup SO^e(2)^*.$

Let $e \in \mathcal{F}^*$, $B$ a differential algebraic subgroup of $\mathfrak{l}_e$ defined over $\mathcal{F}$, and $\beta \in \mathfrak{l}_e$ be such that $\beta + B \in \mathbb{G}_a^m/B$ is rational over $\mathcal{F}$. 
Recall that \( SO^e(2)^\dagger = SO^e(2) \cup SO^e(2)^* \).

Let \( e \in \mathcal{F}^* \), \( B \) a differential algebraic subgroup of \( \mathfrak{l}_e \) defined over \( \mathcal{F} \), and \( \beta \in \mathfrak{l}_e \) be such that \( \beta + B \in \mathbb{G}_a^m/B \) is rational over \( \mathcal{F} \).

The set \( SO^e(B, \beta)^\dagger = \pi_e^{-1}(B) \cup \pi_e^{*-1}(\beta + B) \) is a Zariski dense differential algebraic subgroup of \( SO^e(2)^\dagger \) defined over \( \mathcal{F} \) with \( \pi_e^{-1}(B) = SO^e(B) \) as its component of the identity, and these are all.
Recall that $SO^e(2)^\dagger = SO^e(2) \cup SO^e(2)^*$.

Let $e \in \mathcal{F}^*$, $B$ a differential algebraic subgroup of $\mathfrak{l}_e$ defined over $\mathcal{F}$, and $\beta \in \mathfrak{l}_e$ be such that $\beta + B \in G^m/\mathfrak{b}$ is rational over $\mathcal{F}$.

The set $SO^e(B, \beta)^\dagger = \pi^{-1}_e(B) \cup \pi^{*-1}_e(\beta + B)$ is a Zariski dense differential algebraic subgroup of $SO^e(2)^\dagger$ defined over $\mathcal{F}$ with $\pi^{-1}_e(B) = SO^e(B)$ as its component of the identity, and these are all.

$SO^e(B, \beta)^\dagger$ is conjugate over $\mathcal{F}$ to $SO^e(B', \beta')^\dagger$ if and only if there exists $f \in \mathcal{F}^*$ such that $e'^{-1}e = f^2$, $fB' = B$, and

$$SO^e_{\mathcal{F}}(2) \cap \pi^{-1}_e((f\beta' - \beta + B) \cup (f\beta' + \beta + B)) \neq \emptyset.$$
Example. Let $\mathcal{F} = \mathbb{Q}(i \sin x, i \cos x)$, $\delta = d/dx$. Then $\mathcal{E} = \mathbb{Q}$ and the field of constants of $\mathcal{F}(i)$ is $\mathbb{Q}(i)$ by linear disjointness.
Example. Let $\mathcal{F} = \mathbb{Q}(i \sin x, i \cos x)$, $\delta = d/dx$. Then $\mathcal{C} = \mathbb{Q}$ and the field of constants of $\mathcal{F}(i)$ is $\mathbb{Q}(i)$ by linear disjointness.

Let $e = -1$, $B = 0$, $h = i \sin x$, $g = i \cos x$, and $\beta = \frac{1}{2}$. Then $G_5 = SO^e(0, \beta)^\dagger$ and $G'_5 = SO^e(0, 2\beta)^\dagger$ are both dense differential algebraic subgroups of $SO^e(2)^\dagger$ contained in $\mathbb{P}(\mathcal{F}\mathcal{K})$, but neither is conjugate over $\mathcal{F}$ to $SO^e_{\mathcal{K}}(2)^\dagger = SO^e(0, 0)^\dagger$, which is the union of

$$\text{Ker } \pi_e = \left\{ \begin{pmatrix} \alpha & e\gamma \\ \gamma & \alpha \end{pmatrix} \Bigg| \alpha \in \mathcal{K}, \gamma \in \mathcal{U}, \alpha^2 - e\gamma^2 = 1 \right\}$$

and

$$\text{Ker } \pi^*_e = \left\{ \begin{pmatrix} \alpha & -e\gamma \\ \gamma & -\alpha \end{pmatrix} \Bigg| \alpha \in \mathcal{K}, \gamma \in \mathcal{U}, \alpha^2 - e\gamma^2 = -1 \right\}.$$
If $H$ is infinite, then $H$ is conjugate over $\mathcal{F}$ to one of the following:

1. $G_1 = s\text{SL}_{\mathcal{K}}(2)s^{-1}$, where $s \in \mathbf{P}(\mathcal{F} \mathcal{C}_a)$ and $\ell \Delta s \in \text{sl}_{\mathcal{F}}(2)^m$.

2. $G_2 = \text{ST}(B, \mathcal{K}^*, 0)$, where $B$ is a nonzero differential algebraic subgroup of $\mathcal{G}_a$ defined over $\mathcal{F}$, and $B$, as a vector space over $\mathcal{K}$, has a finite basis in $\mathcal{F}$.

3. $G_3 = \text{ST}(B, \mathcal{P}_{2n}, 0)$, where $n \in \mathbb{N}$, and $B$ is as above.

4. $G_4 = \text{SO}_{\mathcal{K}}^e(2)$, where $e \in \mathcal{C}^*$.

5. $G_5 = \text{SO}^e(0, \beta)^\dagger$ where $e \in \mathcal{C}^*$, $\beta = \left(\frac{1}{2} \frac{\delta_1 h}{g}, \ldots, \frac{1}{2} \frac{\delta_m h}{g}\right)$ for some $h, g \in \mathcal{F}$, $g \neq 0$, and $h^2 - eg^2 \in \mathcal{C}^*$.
If $H$ is finite and if $\mathcal{C}$ is algebraically closed, then $H \subset P(\mathcal{K})$, and $H$ is conjugate over $\mathcal{F}$ to one of the following:

1. $G_6 = C(2n)$, where $n \in \mathbb{N}$.
2. $G_7 = D(4n, e)$, where $n \in \mathbb{N}$, $e \in \mathcal{F}^*$.
3. $G_8 = D(8, e, f, g)$, where $e, f, g \in \mathcal{F}$, $eg \neq 0$, and $f^2 + eg^2 = 1$.
4. $G_9 = T$
5. $G_{10} = OC$
6. $G_{11} = DI$.

\[
D(4n, e) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a^n = 1 \right\} \cup \left\{ \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix} \mid b^{2n} = e^n \right\}
\]

\[
D(8, e, f, g) = \left\{ \pm 1, \pm \begin{pmatrix} 0 & \epsilon \\ -\epsilon^{-1} & 0 \end{pmatrix}, \pm \begin{pmatrix} \alpha & e\gamma \\ \gamma & -\alpha \end{pmatrix}, \pm \begin{pmatrix} -\epsilon\gamma & \epsilon\alpha \\ \epsilon^{-1}\alpha & \epsilon\gamma \end{pmatrix} \right\}
\]

$\epsilon^2 = e$, $\alpha^2 + e\gamma^2 = -1$, $\alpha^2 - e\gamma^2 = f$, $2\alpha\gamma = g$
Example of $D(8, e, f, g)$

- Let $\mathcal{F} = \mathbb{C}(\sin^2 x, \sin x \cos x), \delta/\delta x$. 
Example of $D(8, e, f, g)$

♦ Let $\mathcal{F} = \mathbb{C}(\sin^2 x, \sin x \cos x), \delta/dx$.

♦ Let $e = \sec^2 x$, $f = i \tan x$, and $g = 1$. 
Example of $D(8, e, f, g)$

Let $\mathcal{F} = \mathbb{C}(\sin^2 x, \sin x \cos x), \delta / dx$.

Let $e = \sec^2 x$, $f = i \tan x$, and $g = 1$.

Let $\epsilon = \sec x$, $\alpha = \left(\frac{-1 + i \tan x}{2}\right)^{1/2}$, $\gamma = \frac{1}{2} \alpha^{-1}$. 
Example of $D(8, e, f, g)$

- Let $\mathcal{F} = \mathbb{C}(\sin^2 x, \sin x \cos x), \delta/dx$.
- Let $e = \sec^2 x, f = i \tan x, \text{and } g = 1$.
- Let $\epsilon = \sec x, \alpha = (\frac{-1+i\tan x}{2})^{1/2}, \gamma = \frac{1}{2} \alpha^{-1}$.
- Then if $H = D(8, e, f, g)$, $H$ is not conjugate over $\mathcal{F}$ to $D(8, e)$.

\[
D(4n, e) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a^n = 1 \right\} \cup \left\{ \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix} \mid b^{2n} = e^n \right\}
\]

\[
D(8, e, f, g) = \left\{ \pm 1, \pm \begin{pmatrix} 0 & \epsilon \\ -\epsilon^{-1} & 0 \end{pmatrix}, \pm \begin{pmatrix} \alpha & e\gamma \\ \gamma & -\alpha \end{pmatrix}, \pm \begin{pmatrix} -\epsilon\gamma & \epsilon\alpha \\ \epsilon^{-1}\alpha & \epsilon\gamma \end{pmatrix} \right\}
\]

$\epsilon^2 = e, \quad \alpha^2 + e\gamma^2 = -1, \quad \alpha^2 - e\gamma^2 = f, \quad 2\alpha\gamma = g$
If $\mathcal{E}$ is a differential subfield of $\mathcal{G} = \mathcal{F}\langle t \rangle$ such that $\mathcal{G}$ is strongly normal over $\mathcal{E}$ and $G(\mathcal{G}/\mathcal{E})$ is infinite, then $\mathcal{E}$ is conjugate over $\mathcal{F}$ to one of the following:

1. $\mathcal{E}_1 = \mathcal{F}\langle S\delta_1(t^s), \ldots, S\delta_m(t^s) \rangle$, where $s \in \mathcal{P}(\mathcal{F}\mathcal{C}_a)$ and $\ell\Delta(s) \in \text{sl}_F(2)^m$.
2. $\mathcal{E}_2 = \mathcal{F}\langle (\ell\delta_i L(t))_{1 \leq i \leq m, L \in \Gamma(B)} \rangle$, where $B$ is a nonzero differential algebraic subgroup of $\mathcal{G}_a$ defined over $\mathcal{F}$ and $B$ (as a vector space over $\mathcal{K}$) has a finite basis in $\mathcal{F}$, and $\Gamma(B)$ is the set of all non-zero linear homogeneous differential polynomials vanishing on $B$.
3. $\mathcal{E}_3 = \mathcal{F}\langle (L(t)^n)_{L \in \Gamma(B)} \rangle$, where $n > 0 \in \mathbb{N}$, and $B, \Gamma(B)$ as above.
4. $\mathcal{E}_4 = \mathcal{F}\langle (\delta_it)_{1 \leq i \leq m} \rangle$, where $e \in \mathbb{C}^*$.
5. $\mathcal{E}_5 = \mathcal{F}\langle ((\frac{2\delta_it}{e-t^2} - \frac{\delta_ih}{2eg})^2)_{1 \leq i \leq m} \rangle$, where $e \in \mathbb{C}^*$, and $h, g \in \mathcal{F}$, $g \neq 0$, and $h^2 - eg^2 \in \mathbb{C}^*$. 
Strongly Normal Subfields: Finite Galois Group

If \( G(\mathcal{S}/\mathcal{E}) \) is finite, and if \( C \) is algebraically closed, then \( \mathcal{E} \) is conjugate over \( \mathcal{F} \) to one of the following, and \( \mathcal{S} \) is Galois over \( \mathcal{E} \).

1. \( \mathcal{E}_6 = \mathcal{F}\langle t^n \rangle \), where \( n > 0 \in \mathbb{N} \).

2. \( \mathcal{E}_7 = \mathcal{F}\langle t^n + \frac{e^n}{t^n} \rangle \), where \( n > 0 \in \mathbb{N} \), \( e \in \mathcal{F}^* \).

3. \( \mathcal{E}_8 = \mathcal{F}\langle \frac{g(t^2 + e)^2}{t(gt^2 - 2ft - eg)} \rangle \), where \( e, f, g \in \mathcal{F} \), \( eg \neq 0 \), and \( f^2 + eg^2 = 1 \).

4. \( \mathcal{E}_9 = \mathcal{F}\langle \frac{t^{12} - 33t^8 - 33t^4 + 1}{(t^4 - 1)^2 t^2} \rangle \).

5. \( \mathcal{E}_{10} = \mathcal{F}\langle \frac{(t^{12} - 33t^8 - 33t^4 + 1)^2}{(t^4 - 1)^4 t^4} \rangle \).

6. \( \mathcal{E}_{11} = \mathcal{F}\langle \frac{(t^{30} - 228t^{15} + 494t^{10} + 228t^5 + 1)^3}{t^5(t^{10} + 11t^5 - 1)^5} \rangle \).
The Differential Galois Group for $ST(B, \mathcal{K}^*, 0)$

\[ G_2 = ST(B, \mathcal{K}^*, 0) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \bigg| a \in \mathcal{K}^*, b \in B \right\} \]

where $B$ is a nonzero differential algebraic subgroup of $\mathbb{G}_a$ defined over $\mathcal{F}$, and $B$, as a vector space over $\mathcal{K}$, has a finite basis in $\mathcal{F}$. 


The Differential Galois Group for $ST(B, \mathcal{K}^*, 0)$

$G_2 = ST(B, \mathcal{K}^*, 0) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right| a \in \mathcal{K}^*, b \in B \right\}$ where $B$ is a nonzero differential algebraic subgroup of $G_a$ defined over $\mathcal{F}$, and $B$, as a vector space over $\mathcal{K}$, has a finite basis in $\mathcal{F}$.

$\mathcal{E}_2 = \mathcal{F}\langle (\ell \delta_i L(t))_{1 \leq i \leq m, L \in \Gamma(B)} \rangle$, where $\Gamma(B)$ is the set of all non-zero linear homogeneous differential polynomials vanishing on $B$. ($\Gamma(B)$ can be replaced by a finite subset.)
The Differential Galois Group for $ST(B, \mathcal{K}^*, 0)$

$G_2 = ST(B, \mathcal{K}^*, 0) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right| a \in \mathcal{K}^*, b \in B \right\}$ where $B$ is a nonzero differential algebraic subgroup of $G_a$ defined over $\mathcal{F}$, and $B$, as a vector space over $\mathcal{K}$, has a finite basis in $\mathcal{F}$.

$E_2 = \mathcal{F}\langle (\ell \delta_i L(t))_{1 \leq i \leq m, L \in \Gamma(B)} \rangle$, where $\Gamma(B)$ is the set of all non-zero linear homogeneous differential polynomials vanishing on $B$. ($\Gamma(B)$ can be replaced by a finite subset.)

$G_2$ is $C$-isomorphic to the group $G(\ell, 0)$ of $(\ell + 1) \times (\ell + 1)$ matrices of the form

$$g(b_1, \ldots, b_\ell; a) = \begin{pmatrix}
1 & 0 & \cdots & 0 & b_1 \\
0 & 1 & \cdots & 0 & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & b_\ell \\
0 & 0 & \cdots & 0 & a
\end{pmatrix}$$

where $\ell = \dim \mathcal{K}B$, and $b_1, \ldots, b_\ell, a \in \mathcal{K}$.
\[ G_3 = ST(B, \mathbb{P}_{2n}, 0) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\} \mid a^{2n} = 1, b \in B \] where \( B \) is a nonzero differential algebraic subgroup of \( \mathbb{G}_a \) defined over \( \mathcal{F} \), and \( B \), as a vector space over \( \mathcal{K} \), has a finite basis in \( \mathcal{F} \).
The Differential Galois Group for $ST(B, \mathbb{P}_{2n}, 0)$

- $G_3 = ST(B, \mathbb{P}_{2n}, 0) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a^{2n} = 1, b \in B \right\}$ where $B$ is a nonzero differential algebraic subgroup of $\mathbb{G}_a$ defined over $\mathcal{F}$, and $B$, as a vector space over $\mathcal{K}$, has a finite basis in $\mathcal{F}$.

- $\mathcal{E}_3 = \mathcal{F}\langle (L(t)^n)_{L \in \Gamma(B)} \rangle$, where $n > 0 \in \mathbb{N}$, and $B, \Gamma(B)$ as before. ($\Gamma(B)$ can be replaced by a finite subset.)
The Differential Galois Group for $ST(B, P_{2n}, 0)$

$G_3 = ST(B, P_{2n}, 0) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \Big| a^{2n} = 1, b \in B \right\}$ where $B$ is a nonzero differential algebraic subgroup of $G_a$ defined over $F$, and $B$, as a vector space over $K$, has a finite basis in $F$.

$E_3 = F\langle (L(t)^n)_{L \in \Gamma(B)} \rangle$, where $n > 0 \in \mathbb{N}$, and $B, \Gamma(B)$ as before. ($\Gamma(B)$ can be replaced by a finite subset.)

$G_3$ is $C$-isomorphic to the group $G(\ell, 0)$ of $(\ell + 1) \times (\ell + 1)$ matrices of the form

$$g(b_1, \ldots, b_\ell; a) = \begin{pmatrix} 1 & 0 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_\ell \\ 0 & 0 & \cdots & 0 & a \end{pmatrix}$$

where $\ell = \dim_K B$, and $b_1, \ldots, b_\ell, a^{2n} = 1, a \neq 0$. 
Open Questions

• $G_1 = sSL_K(2)s^{-1}$, where $s \in P(FC_a)$ and $\ell \Delta s \in sl_F(2)^m$. 

• Is $G/E_1$ a Picard-Vessiot Extension? If so, what is a representation of $G_4$ as a $C$-group?

• Ditto: $G_4 = SO_e K(2), e \in C^\ast$.

• Ditto: $G_5 = SO_e(0, \beta)^\dagger, e \in C^\ast, \beta = (\frac{1}{2} \delta_1 h, \ldots, \frac{1}{2} \delta_m h)$ for some $h, g \in F$, $g \neq 0$, and $h^2 - eg^2 \in C^\ast$.

• $E_4 = F\langle \delta_i t \rangle_{1 \leq i \leq m}, e \in C^\ast$, where $e \in C^\ast$.
Open Questions

- $G_1 = sSL_K(2)s^{-1}$, where $s \in P(FC_a)$ and $\ell \Delta s \in sl_F(2)^m$.
- $E_1 = F\langle S\delta_1(t^s), \ldots, S\delta_m(t^s) \rangle$.
Open Questions

=G_1 = sSL_K(2)s^{-1}, where s \in \mathbf{P}(\mathcal{F}C_a) and \ell \Delta s \in sl_K(2)^m.

\mathcal{E}_1 = \mathcal{F}\langle S\delta_1(t^s), \ldots, S\delta_m(t^s)\rangle

Is \mathcal{G}/\mathcal{E}_1 a Picard-Vessiot Extension? If so, what is a representation of \mathcal{G}_4 as a \mathcal{C}-group?
Open Questions

• $G_1 = s SL_\mathcal{K}(2)s^{-1}$, where $s \in P(\mathcal{F}C_\alpha)$ and $\ell \Delta s \in sl_\mathcal{F}(2)^m$.

• $\mathcal{E}_1 = \mathcal{F}\langle S\delta_1(t^s), \ldots, S\delta_m(t^s) \rangle$

• Is $\mathcal{G}/\mathcal{E}_1$ a Picard-Vessiot Extension? If so, what is a representation of $G_4$ as a $C$-group?

• Ditto: $G_4 = SO_{2e}(2)$, $e \in C^*$
Open Questions

• $G_1 = sSL_{\mathcal{K}}(2)s^{-1}$, where $s \in \mathcal{P}(\mathcal{F}\mathcal{C}_a)$ and $\ell \Delta s \in sl_\mathcal{F}(2)^m$.

• $\mathcal{E}_1 = \mathcal{F}\langle S\delta_1(t^s), \ldots, S\delta_m(t^s) \rangle$

• Is $\mathcal{G}/\mathcal{E}_1$ a Picard-Vessiot Extension? If so, what is a representation of $G_4$ as a $\mathcal{C}$-group?

• Ditto: $G_4 = SO^e_{\mathcal{K}}(2)$, $e \in \mathcal{C}^*$

• $\mathcal{E}_4 = \mathcal{F}\langle (\frac{\delta_it}{e-t^2})_{1 \leq i \leq m} \rangle$, where $e \in \mathcal{C}^*$. 
Open Questions

• $G_1 = sSL\mathcal{K}(2)s^{-1}$, where $s \in \mathcal{P}(\mathcal{F}C_a)$ and $\ell \Delta s \in sl\mathcal{F}(2)^m$.

• $\mathcal{E}_1 = \mathcal{F}\langle S\delta_1(t^s), \ldots, S\delta_m(t^s) \rangle$

• Is $\mathcal{G}/\mathcal{E}_1$ a Picard-Vessiot Extension? If so, what is a representation of $G_4$ as a $\mathcal{C}$-group?

• Ditto: $G_4 = SO_e\mathcal{K}(2)$, $e \in \mathcal{C}^*$

• $\mathcal{E}_4 = \mathcal{F}\langle \left(\frac{\delta_i t}{e-t^2}\right)_{1 \leq i \leq m} \rangle$, where $e \in \mathcal{C}^*$.

• Ditto: $G_5 = SO_e(0, \beta)^\dagger$, $e \in \mathcal{C}^*$, $\beta = \left(\frac{1}{2} \frac{\delta_1 h g}{g}, \ldots, \frac{1}{2} \frac{\delta_m h g}{g}\right)$ for some $h, g \in \mathcal{F}$, $g \neq 0$, and $h^2 - eg^2 \in \mathcal{C}^*$. 
Open Questions

- \( G_1 = sSL_\mathcal{K}(2)s^{-1} \), where \( s \in \mathbf{P}(\mathcal{F}_\mathcal{K} \mathcal{C}_a) \) and \( \ell \Delta s \in sl_\mathcal{F}(2)^m \).
- \( \mathcal{E}_1 = \mathcal{F}\langle S\delta_1(t^s), \ldots, S\delta_m(t^s) \rangle \)
- Is \( \mathcal{G}/\mathcal{E}_1 \) a Picard-Vessiot Extension? If so, what is a representation of \( G_4 \) as a \( \mathcal{C} \)-group?
- Ditto: \( G_4 = SO_{\mathcal{K}}^e(2), e \in \mathcal{C}^* \)
- \( \mathcal{E}_4 = \mathcal{F}\langle \left( \frac{\delta_i t}{e-t^2} \right)_{1 \leq i \leq m} \rangle \), where \( e \in \mathcal{C}^* \).
- Ditto: \( G_5 = SO_{\mathcal{K}}^e(0, \beta)^\dagger, e \in \mathcal{C}^*, \beta = \left( \frac{1}{2} \frac{\delta_1 h}{g}, \ldots, \frac{1}{2} \frac{\delta_m h}{g} \right) \) for some \( h, g \in \mathcal{F}, g \neq 0 \), and \( h^2 - eg^2 \in \mathcal{C}^* \).
- \( \mathcal{E}_5 = \mathcal{F}\langle \left( \left( \frac{2\delta_i t}{e-t^2} - \frac{\delta_i h}{2eg} \right)^2 \right)_{1 \leq i \leq m} \rangle \), where \( e \in \mathcal{C}^* \), and \( h, g \in \mathcal{F}, g \neq 0 \), and \( h^2 - eg^2 \in \mathcal{C}^* \).