A Differential Algebra Approach to Linear Boundary Problems

Markus Rosenkranz
\langle\text{M.Rosenkranz@kent.ac.uk}\rangle

School of Mathematics, Statistics and Actuarial Science
University of Kent, United Kingdom

Kolchin Seminar in Differential Algebra
7 July 2014

We acknowledge support from EPSRC First Grant EP/I037474/1.
Overview

Abstract Boundary Problems: Joint work with G. Regensburger [AMPA09] and N. Phisanbut [CASC13].


Markus Rosenkranz
Differential Algebra for Boundary Problems
Abstract Boundary Problems:
Joint work with G. Regensburger [AMPA09]
and N. Phisanbut [CASC13].
Abstract Boundary Problems:
Joint work with G. Regensburger [AMPA09] and N. Phisanbut [CASC13].

Ordinary Integro-Differential Operators:
Overview

1. **Abstract Boundary Problems:**
   Joint work with G. Regensburger [AMPA09] and N. Phisanbut [CASC13].

2. **Ordinary Integro-Differential Operators:**

3. **Partial Integro-Differential Operators:**
   Beginnings with G. Regensburger and L. Tec in [CASC09]. New development with N. Phisanbut [CASC13]. Ongoing work.
Classical Beam Deflection

Thin beam, plane cross sections
Elastic modulus $E$, Moment of area $I$

Normalized horizontal coordinate $x \in [0, 1]$
Deflection $u(x)$, Load $q(x)$
Classical Beam Deflection

Thin beam, plane cross sections
Elastic modulus $E$, Moment of area $I$

Normalized horizontal coordinate $x \in [0, 1]$
Deflection $u(x)$, Load $q(x)$

Euler-Bernoulli Equation: \[
\frac{d^2}{dx^2} \left( EI \frac{d^2 u}{dx^2} \right) = q(x)
\]
Classical Beam Deflection

Thin beam, plane cross sections
Elastic modulus $E$, Moment of area $I$

Normalized horizontal coordinate $x \in [0, 1]$
Deflection $u(x)$, Load $q(x)$

Euler-Bernoulli Equation: $\frac{d^2}{dx^2}(EI \frac{d^2u}{dx^2}) = q(x)$

Simply supported left/right end: $u(0) = u''(0) = 0$ and $u(1) = u''(1) = 0$
Classical Beam Deflection

Thin beam, plane cross sections
Elastic modulus $E$, Moment of area $I$

Normalized horizontal coordinate $x \in [0, 1]$
Deflection $u(x)$, Load $q(x)$

**Euler-Bernoulli Equation:**
\[
\frac{d^2}{dx^2}(EI \frac{d^2u}{dx^2}) = q(x)
\]

Simply supported left/right end: $u(0) = u''(0) = 0$ and $u(1) = u''(1) = 0$
[Free left end: $u'''(0) = u''(0) = 0$]
Classical Beam Deflection

Thin beam, plane cross sections
Elastic modulus $E$, Moment of area $I$

Normalized horizontal coordinate $x \in [0, 1]$
Deflection $u(x)$, Load $q(x)$

Euler-Bernoulli Equation: \[
\frac{d^2}{dx^2} \left( EI \frac{d^2 u}{dx^2} \right) = q(x)
\]

Simply supported left/right end: $u(0) = u''(0) = 0$ and $u(1) = u''(1) = 0$
[Free left end: $u'''(0) = u''(0) = 0$]

Hypothesis: Homogeneous beam, $f \triangleq q/(EI)$
Classical Beam Deflection

Thin beam, plane cross sections
Elastic modulus $E$, Moment of area $I$

Normalized horizontal coordinate $x \in [0, 1]$
Deflection $u(x)$, Load $q(x)$

Euler-Bernoulli Equation: \[
\frac{d^2}{dx^2}(EI \frac{d^2u}{dx^2}) = q(x)
\]

Simply supported left/right end: $u(0) = u''(0) = 0$ and $u(1) = u''(1) = 0$
[Free left end: $u'''(0) = u''(0) = 0$]

Hypothesis: Homogeneous beam, $f \triangleq q/(EI)$

Boundary Problem:

\[
\begin{align*}
    u''' &= f \\
    u(0) &= u''(0) = u(1) = u''(1) = 0
\end{align*}
\]
Classical Beam Deflection

Thin beam, plane cross sections
Elastic modulus $E$, Moment of area $I$

Normalized horizontal coordinate $x \in [0, 1]$
Deflection $u(x)$, Load $q(x)$

Euler-Bernoulli Equation:
$$\frac{d^2}{dx^2}(EI \frac{d^2u}{dx^2}) = q(x)$$

Simply supported left/right end: $u(0) = u''(0) = 0$ and $u(1) = u''(1) = 0$
[Free left end: $u'''(0) = u''(0) = 0$]

Hypothesis: Homogeneous beam, $f \triangleq q/(EI)$

Boundary Problem:

\[
\begin{align*}
    u''' &= f \\
    u(0) &= u''(0) = u(1) = u''(1) = 0
\end{align*}
\]

Classically $u \in C^4[0, 1]$. 
Analytic Method

Superposition Principle:

Total deflection as superposition of $u_\xi(x)$ weighted by $f(\xi)$

Hence $u(x) = \int_0^1 g(x,\xi) f(\xi) d\xi$

Green's function $g(x,\xi) \equiv u_\xi(x)$

Solution for simply supported Euler-Bernoulli beam:

$$g(x,\xi) = \begin{cases} 
\frac{1}{3} x \xi - \frac{1}{6} \xi^3 - \frac{1}{2} x^2 \xi + \frac{1}{6} x \xi^3 + \frac{1}{6} x^3 \xi & \text{für } 0 \leq \xi \leq x \leq 1, \\
\frac{1}{3} x \xi - \frac{1}{2} x \xi^2 - \frac{1}{6} x^3 & \text{für } 0 \leq x \leq \xi \leq 1 
\end{cases}$$

Question: How does differential algebra help in finding this solution?
Superposition Principle:
- Deflection $u_\xi(x)$ for normalized point loads $f = \delta_\xi$ bei $\xi \in [0, 1]$
Analytic Method

Superposition Principle:

- Deflection $u_\xi(x)$ for normalized point loads $f = \delta_\xi$ bei $\xi \in [0, 1]$
- Total deflection as superposition of $u_\xi(x)$ weighted by $f(\xi)$
Superposition Principle:

- Deflection $u_\xi(x)$ for normalized point loads $f = \delta_\xi$ bei $\xi \in [0, 1]$
- Total deflection as superposition of $u_\xi(x)$ weighted by $f(\xi)$
- Hence $u(x) = \int_0^1 g(x, \xi) f(\xi) \, d\xi$
Analytic Method

Superposition Principle:

- Deflection $u_\xi(x)$ for normalized point loads $f = \delta_\xi$ bei $\xi \in [0, 1]$
- Total deflection as superposition of $u_\xi(x)$ weighted by $f(\xi)$
- Hence $u(x) = \int_0^1 u_\xi(x) f(\xi) \, d\xi$
- Green’s function $g(x, \xi) \triangleq u_\xi(x)$
Analytic Method

Superposition Principle:

- Deflection $u_\xi(x)$ for normalized point loads $f = \delta_\xi$ bei $\xi \in [0, 1]$
- Total deflection as superposition of $u_\xi(x)$ weighted by $f(\xi)$
- Hence $u(x) = \int_0^1 g(x, \xi) f(\xi) \, d\xi$
- Green’s function $g(x, \xi) \triangleq u_\xi(x)$

Solution for simply supported Euler-Bernoulli beam:

$$g(x, \xi) = \begin{cases} 
\frac{1}{3} x \xi - \frac{1}{6} \xi^3 - \frac{1}{2} x^2 \xi + \frac{1}{6} x \xi^3 + \frac{1}{6} x^3 \xi & \text{für } 0 \leq \xi \leq x \leq 1, \\
\frac{1}{3} x \xi - \frac{1}{2} x \xi^2 - \frac{1}{6} x^3 + \frac{1}{6} x \xi^3 + \frac{1}{6} x^3 \xi & \text{für } 0 \leq x \leq \xi \leq 1
\end{cases}$$

Question: How does differential algebra help in finding this solution?
Connecting Differential Algebra with Boundary Values

Basic view of differential algebra:

$$F = \mathbb{C}_\infty(0, 1)$$ is a differential ring.

$$\partial: F \to F,$$ $\partial(u + v) = \partial(u) + \partial(v)$ and $\partial(uv) = \partial(u)v + r\partial(v).$

For $u \in F$ we have $u' \triangleq \partial(u) \in F$ but no $u(0)$ or $u'(0)$.

Which other algebraic structure can we find in $$(\mathbb{C}_\infty[0, 1], \partial)$$?

Short answer: Point evaluations = multiplicative linear functionals on $F$. Linked to differential structure via integration (Rota-Baxter ring).

Evaluation/Integration: Two sides of a single coin:

INTEGRO-DIFFERENTIAL ALGEBRA

...and the rest is Linear Algebra.

Markus Rosenkranz

Differential Algebra for Boundary Problems
Basic view of differential algebra: $\mathcal{F} = C^\infty(0, 1)$ is a differential ring.
Basic view of differential algebra: $\mathcal{F} = C^\infty(0, 1)$ is a differential ring.

$\partial: \mathcal{F} \rightarrow \mathcal{F}, \quad \partial(u + v) = \partial(u) + \partial(v)$ and $\partial(uv) = \partial(u)v + r\partial(v)$
Basic view of differential algebra: $\mathcal{F} = C^\infty(0,1)$ is a differential ring.

$\partial: \mathcal{F} \to \mathcal{F}$, \( \partial(u + v) = \partial(u) + \partial(v) \) and \( \partial(uv) = \partial(u)v + r\partial(v) \)

For \( u \in \mathcal{F} \) we have \( u' \triangleq \partial(u) \in \mathcal{F} \) but no \( u(0) \) or \( u'(0) \).
Basic view of differential algebra: $\mathcal{F} = C^\infty(0, 1)$ is a differential ring.

$$\partial: \mathcal{F} \to \mathcal{F}, \quad \partial(u + v) = \partial(u) + \partial(v) \text{ and } \partial(uv) = \partial(u)v + r\partial(v)$$

For $u \in \mathcal{F}$ we have $u' \triangleq \partial(u) \in \mathcal{F}$ but no $u(0)$ or $u'(0)$. Which other algebraic structure can we find in $(C^\infty[0, 1], \partial)$?
Basic view of differential algebra: $\mathcal{F} = C^\infty(0, 1)$ is a differential ring.

$\partial: \mathcal{F} \rightarrow \mathcal{F}, \quad \partial(u + v) = \partial(u) + \partial(v) \text{ and } \partial(uv) = \partial(u)v + r\partial(v)$

For $u \in \mathcal{F}$ we have $u' \triangleq \partial(u) \in \mathcal{F}$ but no $u(0)$ or $u'(0)$.

Which other algebraic structure can we find in $(C^\infty[0, 1], \partial)$?

Short answer:

- Point evaluations = multiplicative linear functionals on $\mathcal{F}$. 

Linked to differential structure via integration (Rota-Baxter ring).

Evaluation/Integration: Two sides of a single coin:

[INTEGRO-DIFFERENTIAL ALGEBRA]

...and the rest is Linear Algebra.
Basic view of differential algebra: $\mathcal{F} = C^\infty(0, 1)$ is a differential ring.

$$\partial: \mathcal{F} \rightarrow \mathcal{F}, \quad \partial(u + v) = \partial(u) + \partial(v) \text{ and } \partial(uv) = \partial(u)v + r\partial(v)$$

For $u \in \mathcal{F}$ we have $u' \triangleq \partial(u) \in \mathcal{F}$ but no $u(0)$ or $u'(0)$.

Which other algebraic structure can we find in $(C^\infty[0, 1], \partial)$?

Short answer:

- Point evaluations = multiplicative linear functionals on $\mathcal{F}$.
- Linked to differential structure via integration (Rota-Baxter ring).
Basic view of differential algebra: $\mathcal{F} = \mathcal{C}^\infty(0, 1)$ is a differential ring.

\[ \partial: \mathcal{F} \to \mathcal{F}, \quad \partial(u + v) = \partial(u) + \partial(v) \text{ and } \partial(uv) = \partial(u)v + r\partial(v) \]

For $u \in \mathcal{F}$ we have $u' \triangleq \partial(u) \in \mathcal{F}$ but no $u(0)$ or $u'(0)$.

Which other algebraic structure can we find in $(\mathcal{C}^\infty[0, 1], \partial)$?

Short answer:

- Point evaluations = multiplicative linear functionals on $\mathcal{F}$.
- Linked to differential structure via integration (Rota-Baxter ring).
- Evaluation/Integration: Two sides of a single coin:
Basic view of differential algebra: \( \mathcal{F} = \mathcal{C}^\infty(0, 1) \) is a differential ring.

\[ \partial: \mathcal{F} \to \mathcal{F}, \quad \partial(u + v) = \partial(u) + \partial(v) \quad \text{and} \quad \partial(uv) = \partial(u)v + r\partial(v) \]

For \( u \in \mathcal{F} \) we have \( u' \triangleq \partial(u) \in \mathcal{F} \) but no \( u(0) \) or \( u'(0) \).

Which other algebraic structure can we find in \( (\mathcal{C}^\infty[0, 1], \partial) \)?

Short answer:

- Point evaluations = multiplicative linear functionals on \( \mathcal{F} \).
- Linked to differential structure via integration (Rota-Baxter ring).
- Evaluation/Integration: Two sides of a single coin:

\[ \text{INTEGRO-DIFFERENTIAL ALGEBRA} \]
Connecting Differential Algebra with Boundary Values

Basic view of differential algebra: \( \mathcal{F} = C^\infty(0, 1) \) is a differential ring.

\[ \partial: \mathcal{F} \rightarrow \mathcal{F}, \quad \partial(u + v) = \partial(u) + \partial(v) \text{ and } \partial(uv) = \partial(u)v + r\partial(v) \]

For \( u \in \mathcal{F} \) we have \( u' \triangleq \partial(u) \in \mathcal{F} \) but no \( u(0) \) or \( u'(0) \).

Which other algebraic structure can we find in \( (C^\infty[0, 1], \partial) \)?

Short answer:

- Point evaluations = multiplicative linear functionals on \( \mathcal{F} \).
- Linked to differential structure via integration (Rota-Baxter ring).
- Evaluation/Integration: Two sides of a single coin:

\[ \text{INTEGRO-DIFFERENTIAL ALGEBRA} \]

... and the rest is Linear Algebra.
Outline

1 Abstract Boundary Problems

2 Ordinary Integro-Differential Operators

3 Partial Integro-Differential Operators

4 Conclusion
Let $F$, $G$ be fixed (infinite-dimensional) vector spaces.

**Definition**

An (abstract) boundary problem is a pair $(T, B)$ where $T: F \to G$ is an epimorphism and $B \leq F^*$ is orthogonally closed.

We call $T$ the "differential operator" and $B$ the "boundary space" of the problem.

A Galois connection $P(F) \leftrightarrow \bar{P}(F^*)$

$A \leq F \mapsto A^\bot := \{ \beta \in F^* \mid \beta(f) = 0 \text{ for all } f \in A \}$

$B \leq F^* \mapsto B^\bot := \{ f \in F \mid \beta(f) = 0 \text{ for all } \beta \in B \}$

We call $B \leq F^*$ orthogonally closed if $B^\bot^\bot = B$.

Note that all $A \leq F$ are orthogonally closed.

$\bar{P}(F^*) = \text{Orthogonally closed subspaces of } F^*$

Complete complemented modular lattice, isomorphic to $P(F)$

Contains finite dimensional sublattice.
Let $\mathcal{F}, \mathcal{G}$ be fixed (infinite-dimensional) vector spaces.
Abstract Boundary Problems

Let $\mathcal{F}, \mathcal{G}$ be fixed (infinite-dimensional) vector spaces.

**Definition**

An (abstract) **boundary problem** is a pair $(T, B)$ where $T: \mathcal{F} \to \mathcal{G}$ is an epimorphism and $B \leq \mathcal{F}^*$ is orthogonally closed.
Abstract Boundary Problems

Let $\mathcal{F}, \mathcal{G}$ be fixed (infinite-dimensional) vector spaces.

**Definition**

An (abstract) **boundary problem** is a pair $(T, \mathcal{B})$ where $T : \mathcal{F} \rightarrow \mathcal{G}$ is an epimorphism and $\mathcal{B} \leq \mathcal{F}^*$ is orthogonally closed. We call $T$ the “differential operator” and $\mathcal{B}$ the “boundary space” of the problem.

| $\mathcal{A} \leq \mathcal{F}$ | $\mathcal{A}^\perp := \{ \beta \in \mathcal{F}^* | \beta(f) = 0 \text{ for all } f \in \mathcal{A} \}$ |
|-----------------------------|----------------------------------------------------------------------------------|
| $\mathcal{B} \leq \mathcal{F}^*$ | $\mathcal{B}^\perp := \{ f \in \mathcal{F} | \beta(f) = 0 \text{ for all } \beta \in \mathcal{B} \}$ |

Note that all $\mathcal{A} \leq \mathcal{F}$ are orthogonally closed.

$\overline{\mathcal{P}}(\mathcal{F}^*) = \text{Orthogonally closed subspaces of } \mathcal{F}^*$

Contains finite dimensional sublattice.
Abstract Boundary Problems

Let $\mathcal{F}, \mathcal{G}$ be fixed (infinite-dimensional) vector spaces.

**Definition**

An (abstract) **boundary problem** is a pair $(T, B)$ where $T : \mathcal{F} \to \mathcal{G}$ is an epimorphism and $B \leq \mathcal{F}^*$ is orthogonally closed. We call $T$ the “differential operator” and $B$ the “boundary space” of the problem.

Galois connection $\mathbb{P}(\mathcal{F}) \leftrightarrow \overline{\mathbb{P}}(\mathcal{F}^*)$
Let \( \mathcal{F}, \mathcal{G} \) be fixed (infinite-dimensional) vector spaces.

**Definition**

An (abstract) **boundary problem** is a pair \((T, \mathcal{B})\) where \(T : \mathcal{F} \to \mathcal{G}\) is an epimorphism and \(\mathcal{B} \leq \mathcal{F}^*\) is orthogonally closed. We call \(T\) the “differential operator” and \(\mathcal{B}\) the “boundary space” of the problem.

Galois connection \(\mathcal{P}(\mathcal{F}) \leftrightarrow \overline{\mathcal{P}}(\mathcal{F}^*)\)

\[ \mathcal{A} \leq \mathcal{F} \quad \mapsto \quad \mathcal{A}^\perp := \{ \beta \in \mathcal{F}^* \mid \beta(f) = 0 \text{ for all } f \in \mathcal{A} \} \]
Let $\mathcal{F}, \mathcal{G}$ be fixed (infinite-dimensional) vector spaces.

**Definition**

An (abstract) **boundary problem** is a pair $(T, B)$ where $T : \mathcal{F} \to \mathcal{G}$ is an epimorphism and $B \leq \mathcal{F}^\ast$ is orthogonally closed. We call $T$ the “differential operator” and $B$ the “boundary space” of the problem.

Galois connection

$$\mathbb{P}(\mathcal{F}) \leftrightarrow \bar{\mathbb{P}}(\mathcal{F}^\ast)$$

$$\begin{align*}
\mathcal{A} \leq \mathcal{F} & \mapsto \mathcal{A}^\perp := \{ \beta \in \mathcal{F}^\ast \mid \beta(f) = 0 \text{ for all } f \in \mathcal{A} \} \\
\mathcal{B} \leq \mathcal{F}^\ast & \mapsto \mathcal{B}^\perp := \{ f \in \mathcal{F} \mid \beta(f) = 0 \text{ for all } \beta \in \mathcal{B} \}
\end{align*}$$
Let $\mathcal{F}, \mathcal{G}$ be fixed (infinite-dimensional) vector spaces.

**Definition**

An (abstract) **boundary problem** is a pair $(T, \mathcal{B})$ where $T : \mathcal{F} \to \mathcal{G}$ is an epimorphism and $\mathcal{B} \leq \mathcal{F}^*$ is orthogonally closed. We call $T$ the “differential operator” and $\mathcal{B}$ the “boundary space” of the problem.

Galois connection $\mathbb{P}(\mathcal{F}) \leftrightarrow \overline{\mathbb{P}}(\mathcal{F}^*)$

- $\mathcal{A} \leq \mathcal{F} \quad \mapsto \quad \mathcal{A}^\perp := \{ \beta \in \mathcal{F}^* \mid \beta(f) = 0 \text{ for all } f \in \mathcal{A} \}$
- $\mathcal{B} \leq \mathcal{F}^* \quad \mapsto \quad \mathcal{B}^\perp := \{ f \in \mathcal{F} \mid \beta(f) = 0 \text{ for all } \beta \in \mathcal{B} \}$

We call $\mathcal{B} \leq \mathcal{F}^*$ **orthogonally closed** if $\mathcal{B}^{\perp \perp} = \mathcal{B}$. 

Markus Rosenkranz  
Differential Algebra for Boundary Problems
Let $\mathcal{F}, \mathcal{G}$ be fixed (infinite-dimensional) vector spaces.

**Definition**

An (abstract) **boundary problem** is a pair $(T, \mathcal{B})$ where $T: \mathcal{F} \to \mathcal{G}$ is an epimorphism and $\mathcal{B} \leq \mathcal{F}^*$ is orthogonally closed. We call $T$ the “differential operator” and $\mathcal{B}$ the “boundary space” of the problem.

Galois connection $\mathbb{P}(\mathcal{F}) \leftrightarrow \mathcal{P}(\mathcal{F}^*)$

- $\mathcal{A} \leq \mathcal{F} \mapsto \mathcal{A}^\perp := \{ \beta \in \mathcal{F}^* \mid \beta(f) = 0 \text{ for all } f \in \mathcal{A} \}$
- $\mathcal{B} \leq \mathcal{F}^* \mapsto \mathcal{B}^\perp := \{ f \in \mathcal{F} \mid \beta(f) = 0 \text{ for all } \beta \in \mathcal{B} \}$

We call $\mathcal{B} \leq \mathcal{F}^*$ **orthogonally closed** if $\mathcal{B}^\perp \perp = \mathcal{B}$.

Note that all $\mathcal{A} \leq \mathcal{F}$ are orthogonally closed.
Abstract Boundary Problems

Let $\mathcal{F}, \mathcal{G}$ be fixed (infinite-dimensional) vector spaces.

**Definition**

An (abstract) **boundary problem** is a pair $(T, \mathcal{B})$ where $T : \mathcal{F} \to \mathcal{G}$ is an epimorphism and $\mathcal{B} \leq \mathcal{F}^*$ is orthogonally closed. We call $T$ the “differential operator” and $\mathcal{B}$ the “boundary space” of the problem.

Galois connection $\mathcal{P}(\mathcal{F}) \leftrightarrow \bar{\mathcal{P}}(\mathcal{F}^*)$

$\mathcal{A} \leq \mathcal{F} \quad \mapsto \quad \mathcal{A}^\perp := \{ \beta \in \mathcal{F}^* \mid \beta(f) = 0 \text{ for all } f \in \mathcal{A} \}$

$\mathcal{B} \leq \mathcal{F}^* \quad \mapsto \quad \mathcal{B}^\perp := \{ f \in \mathcal{F} \mid \beta(f) = 0 \text{ for all } \beta \in \mathcal{B} \}$

We call $\mathcal{B} \leq \mathcal{F}^*$ **orthogonally closed** if $\mathcal{B}^{\perp \perp} = \mathcal{B}$.

Note that all $\mathcal{A} \leq \mathcal{F}$ are orthogonally closed.

$\bar{\mathcal{P}}(\mathcal{F}^*) = \text{Orthogonally closed subspaces of } \mathcal{F}^*$:
Let $\mathcal{F}, \mathcal{G}$ be fixed (infinite-dimensional) vector spaces.

**Definition**

An (abstract) **boundary problem** is a pair $(T, B)$ where $T : \mathcal{F} \to \mathcal{G}$ is an epimorphism and $B \leq \mathcal{F}^*$ is orthogonally closed. We call $T$ the “differential operator” and $B$ the “boundary space” of the problem.

Galois connection $\mathcal{P}(\mathcal{F}) \leftrightarrow \mathcal{P}(\mathcal{F}^*)$

$A \leq \mathcal{F} \quad \mapsto \quad A^\perp := \{ \beta \in \mathcal{F}^* \mid \beta(f) = 0 \text{ for all } f \in A \}$

$B \leq \mathcal{F}^* \quad \mapsto \quad B^\perp := \{ f \in \mathcal{F} \mid \beta(f) = 0 \text{ for all } \beta \in B \}$

We call $B \leq \mathcal{F}^*$ **orthogonally closed** if $B^{\perp\perp} = B$.

Note that all $A \leq \mathcal{F}$ are orthogonally closed.

$\mathcal{P}(\mathcal{F}^*) = \text{Orthogonally closed subspaces of } \mathcal{F}^*$:

- Complete complemented modular lattice, isomorphic to $\mathcal{P}(\mathcal{F})$
Let $\mathcal{F}, \mathcal{G}$ be fixed (infinite-dimensional) vector spaces.

**Definition**

An (abstract) **boundary problem** is a pair $(T, B)$ where $T : \mathcal{F} \to \mathcal{G}$ is an epimorphism and $B \leq \mathcal{F}^*$ is orthogonally closed. We call $T$ the “differential operator” and $B$ the “boundary space” of the problem.

Galois connection $\bar{\mathbb{P}}(\mathcal{F}) \rightleftarrows \bar{\mathbb{P}}(\mathcal{F}^*)$

$\mathcal{A} \leq \mathcal{F} \quad \mapsto \quad \mathcal{A}^\perp := \{ \beta \in \mathcal{F}^* \mid \beta(f) = 0 \text{ for all } f \in \mathcal{A} \}$

$\mathcal{B} \leq \mathcal{F}^* \quad \mapsto \quad \mathcal{B}^\perp := \{ f \in \mathcal{F} \mid \beta(f) = 0 \text{ for all } \beta \in \mathcal{B} \}$

We call $\mathcal{B} \leq \mathcal{F}^*$ **orthogonally closed** if $\mathcal{B}^\perp\perp = \mathcal{B}$.

Note that all $\mathcal{A} \leq \mathcal{F}$ are orthogonally closed.

$\bar{\mathbb{P}}(\mathcal{F}^*) = \text{Orthogonally closed subspaces of } \mathcal{F}^*$:

- Complete complemented modular lattice, isomorphic to $\mathbb{P}(\mathcal{F})$
- Contains finite dimensional sublattice.
### Definition

A boundary problem \((T, \mathcal{F})\) is called **regular** if \(\mathcal{B}^\perp + \ker T = \mathcal{F}\).

Equivalent to requiring that \(Tu = f\) \(\beta(u) = 0\) \((\beta \in \mathcal{B})\) has a unique solution \(u \in \mathcal{F}\) for every \(f \in \mathcal{G}\).

Hence define **Green's operator** \(G : T \to \mathcal{F}\) by \(Gf = u\).

This means \(TG = 1\) and \(\text{im} G = \mathcal{B}^\perp\).

We write \((T, \mathcal{B})^{-1}\) for \(G\).
A boundary problem \((T, \mathcal{F})\) is called **regular** if \(B^\perp \cap \ker T = \mathcal{F}\).

Equivalent to requiring that

\[
\begin{align*}
Tu &= f \\
\beta(u) &= 0 \quad (\beta \in B)
\end{align*}
\]

has a unique solution \(u \in \mathcal{F}\) for every \(f \in \mathcal{G}\).
A boundary problem \((T, \mathcal{F})\) is called \textit{regular} if \(\mathcal{B} \perp \ker T = \mathcal{F}\).

Equivalent to requiring that

\[
\begin{align*}
Tu &= f \\
\beta(u) &= 0 \quad (\beta \in \mathcal{B})
\end{align*}
\]

has a unique solution \(u \in \mathcal{F}\) for every \(f \in \mathcal{G}\).

Hence define \textit{Green’s operator} \(G: \mathcal{G} \to \mathcal{F}\) by \(Gf = u\).
A boundary problem \((T, \mathcal{F})\) is called **regular** if \(\mathcal{B}^\perp \perp \ker T = \mathcal{F}\).

Equivalent to requiring that

\[
\begin{align*}
Tu &= f \\
\beta(u) &= 0 \quad (\beta \in \mathcal{B})
\end{align*}
\]

has a unique solution \(u \in \mathcal{F}\) for every \(f \in \mathcal{G}\).

Hence define **Green’s operator** \(G : \mathcal{G} \to \mathcal{F}\) by \(Gf = u\).

This means \(TG = 1\) and \(\text{im } G = \mathcal{B}^\perp\).
A boundary problem \((T, \mathcal{F})\) is called **regular** if \(\mathcal{B} \perp \ker T = \mathcal{F}\).

Equivalent to requiring that

\[
\begin{align*}
Tu &= f \\
\beta(u) &= 0 \quad (\beta \in \mathcal{B})
\end{align*}
\]

has a unique solution \(u \in \mathcal{F}\) for every \(f \in \mathcal{G}\).

Hence define **Green’s operator** \(G : \mathcal{G} \to \mathcal{F}\) by \(Gf = u\).

This means \(TG = 1\) and \(\text{im } G = \mathcal{B} \perp\).

We write \((T, \mathcal{B})^{-1}\) for \(G\).
Composition of Boundary Problems

For \((T_1, B_1)\) and \((T_2, B_2)\) with \(\mathcal{F} \xrightarrow{T_2} \mathcal{G} \xrightarrow{T_1} \mathcal{H}\) define

\[
(T_1, B_1) \cdot (T_2, B_2) = (T_1T_2, T_2^*(B_1) + B_2),
\]

which is again a boundary problem.
Composition of Boundary Problems

For \((T_1, \mathcal{B}_1)\) and \((T_2, \mathcal{B}_2)\) with \(\mathcal{F} \xrightarrow{T_2} \mathcal{G} \xrightarrow{T_1} \mathcal{H}\) define

\[(T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2) = (T_1T_2, T_2^*(\mathcal{B}_1) + \mathcal{B}_2),\]

which is again a boundary problem.

**Proposition**

The composition of regular boundary problems is regular, and its Green’s operator is \(G_2G_1\).
Composition of Boundary Problems

For \((T_1, B_1)\) and \((T_2, B_2)\) with \(\mathcal{F} \xrightarrow{T_2} \mathcal{G} \xrightarrow{T_1} \mathcal{H}\) define

\[
(T_1, B_1) \cdot (T_2, B_2) = (T_1T_2, T_2^*(B_1) + B_2),
\]

which is again a boundary problem.

**Proposition**

The composition of regular boundary problems is regular, and its Green’s operator is \(G_2G_1\). In other words, we have

\[
((T_1, B_1) \cdot (T_2, B_2))^{-1} = (T_2, B_2)^{-1} \cdot (T_1, B_1)^{-1}.
\]
Composition of Boundary Problems

For \((T_1, B_1)\) and \((T_2, B_2)\) with \(\mathcal{F} \xrightarrow{T_2} \mathcal{G} \xrightarrow{T_1} \mathcal{H}\) define

\[(T_1, B_1) \cdot (T_2, B_2) = (T_1 T_2, T_2^*(B_1) + B_2),\]

which is again a boundary problem.

**Proposition**

The composition of regular boundary problems is regular, and its Green’s operator is \(G_2 G_1\). In other words, we have

\[\left((T_1, B_1) \cdot (T_2, B_2)\right)^{-1} = (T_2, B_2)^{-1} \cdot (T_1, B_1)^{-1}.\]

Moreover, the sum \(T_2^*(B_1) + B_2\) is direct.
Composition of Boundary Problems

For \((T_1, B_1)\) and \((T_2, B_2)\) with \(F \xrightarrow{T_2} G \xrightarrow{T_1} H\) define

\[
(T_1, B_1) \cdot (T_2, B_2) = (T_1 T_2, T_2^*(B_1) + B_2),
\]

which is again a boundary problem.

**Proposition**

The composition of regular boundary problems is regular, and its Green’s operator is \(G_2 G_1\). In other words, we have

\[
((T_1, B_1) \cdot (T_2, B_2))^{-1} = (T_2, B_2)^{-1} \cdot (T_1, B_1)^{-1}.
\]

Moreover, the sum \(T_2^*(B_1) + B_2\) is direct.

Therefore (for fixed base field):

- All boundary problems form a category \(\text{BnProb}\) with \(F \xrightarrow{(T, B)} G\).
Composition of Boundary Problems

For \((T_1, B_1)\) and \((T_2, B_2)\) with \(\mathcal{F} \xrightarrow{T_2} \mathcal{G} \xrightarrow{T_1} \mathcal{H}\) define

\[(T_1, B_1) \cdot (T_2, B_2) = (T_1 T_2, T_2^*(B_1) + B_2),\]

which is again a boundary problem.

**Proposition**

The composition of regular boundary problems is regular, and its Green’s operator is \(G_2 G_1\). In other words, we have

\[\left((T_1, B_1) \cdot (T_2, B_2)\right)^{-1} = (T_2, B_2)^{-1} \cdot (T_1, B_1)^{-1}.\]

Moreover, the sum \(T_2^*(B_1) + B_2\) is direct.

Therefore (for fixed base field):

- All boundary problems form a category \(\text{BnProb}\) with \(\mathcal{F} \xrightarrow{(T, B)} \mathcal{G}\).
- Regular boundary problems subcategory \(\text{BnProb}^*\).
Composition of Boundary Problems

For \((T_1, B_1)\) and \((T_2, B_2)\) with \(\mathcal{F} \xrightarrow{T_2} \mathcal{G} \xrightarrow{T_1} \mathcal{H}\) define

\[(T_1, B_1) \cdot (T_2, B_2) = (T_1 T_2, T_2^*(B_1) + B_2),\]

which is again a boundary problem.

**Proposition**

The composition of regular boundary problems is regular, and its Green’s operator is \(G_2 G_1\). In other words, we have

\[
((T_1, B_1) \cdot (T_2, B_2))^{-1} = (T_2, B_2)^{-1} \cdot (T_1, B_1)^{-1}.
\]

Moreover, the sum \(T_2^*(B_1) + B_2\) is direct.

Therefore (for fixed base field):

- All boundary problems form a category \(\text{BnProb}\) with \(\mathcal{F} \xrightarrow{(T,B)} \mathcal{G}\).
- Regular boundary problems subcategory \(\text{BnProb}^*\).
- Monoids \(\text{BnProb}(\mathcal{F})\) and \(\text{BnProb}^*(\mathcal{F})\).
A dual problem is a pair \((S, G)\) where \(G: \mathcal{G} \to \mathcal{F}\) is a monomorphism and \(S \leq \mathcal{F}\) is arbitrary.
**Definition**

A **dual problem** is a pair \((S, G)\) where \(G: G \to \mathcal{F}\) is a monomorphism and \(S \leq \mathcal{F}\) is arbitrary. It is **regular** if \(S \oplus \text{im } G = \mathcal{F}\).
A dual problem is a pair \((S, G)\) where \(G: G \rightarrow F\) is a monomorphism and \(S \leq F\) is arbitrary. It is regular if \(S + \text{im } G = F\).

Green’s operator \(T := (S, G)^{-1}\) defined by \(TG = 1\), \(\ker T = S\).
A **dual problem** is a pair \((S, G)\) where \(G: \mathcal{G} \to \mathcal{F}\) is a monomorphism and \(S \leq \mathcal{F}\) is arbitrary. It is **regular** if \(S + \text{im } G = \mathcal{F}\).

Green’s operator \(T := (S, G)^{-1}\) defined by \(TG = 1, \ker T = S\).
Dual composition \((K_2, G_2) \cdot (K_1, G_1) = (K_2 + G_2(K_1), G_2G_1)\).
A dual problem is a pair \((S, G)\) where \(G: \mathcal{G} \rightarrow \mathcal{F}\) is a monomorphism and \(S \leq \mathcal{F}\) is arbitrary. It is regular if \(S + \text{im } G = \mathcal{F}\).

Green's operator \(T := (S, G)^{-1}\) defined by \(TG = 1, \ker T = S\).

Dual composition \((K_2, G_2) \cdot (K_1, G_1) = (K_2 + G_2(K_1), G_2G_1)\).

Categories \(\text{DuProb}\) and \(\text{DuProb}^*\).
**Definition**

A **dual problem** is a pair \((S, G)\) where \(G : \mathcal{G} \rightarrow \mathcal{F}\) is a monomorphism and \(S \leq \mathcal{F}\) is arbitrary. It is **regular** if \(S + \text{im} G = \mathcal{F}\).

Green’s operator \(T := (S, G)^{-1}\) defined by \(TG = 1, \ker T = S\).

Dual composition \((K_2, G_2) \cdot (K_1, G_1) = (K_2 + G_2(K_1), G_2G_1)\).

Categories **DuProb** and **DuProb**\(^*\).

**Proposition**

The contravariant functor \((T, \mathcal{F}) \mapsto (\ker T, (T, B)^{-1})\) together with its inverse \((S, G) \mapsto ((S, G)^{-1}, \text{im} \perp G)\) establishes an isomorphism of categories **BnProb**\(^*\) \(\cong (\text{DuProb}^*)^{\text{op}}\).
Proposition
For a regular boundary problem \((T, F)\), the Green's operator is given by
\[ G = (1 - P) T \]
where \(P\) is the projector onto \(\ker T\) along \(B\|\) and \(T\) is an arbitrary right inverse of \(T\).

For a regular dual problem \((S, G)\), the Green's operator is given by
\[ T = G \cdot (1 - P) \]
where \(P\) is the projector onto \(S\) along \(\text{im } G\) and \(G\) is an arbitrary left inverse of \(G\).

If \(\dim B < \infty\) or \(\dim S < \infty\) then:
\[ B = [\beta_1, \ldots, \beta_n] \]
and
\[ \ker T = [u_1, \ldots, u_n] : \]

Regularity \(\iff\) Evaluation matrix
\[ \beta(u) = [\beta_i(u_j)] \in \text{GL}_n(K) \]

Projector
\[ P = u \cdot \beta(u) - 1 \cdot \beta \]

Analogous for dual problem:
\[ \text{im } \perp G = [\beta_1, \ldots, \beta_n] \]
and
\[ S = [u_1, \ldots, u_n] \]
Proposition

For a regular boundary problem \((T, \mathcal{F})\), the Green’s operator is given by
\[ G = (1 - P)T\Diamond \]
where \(P\) is the projector onto \(\ker T\) along \(B^\perp\) and \(T\Diamond\) is an arbitrary right inverse of \(T\).
Determination of Green’s Operators

**Proposition**

For a regular boundary problem \((T, \mathcal{F})\), the Green’s operator is given by \(G = (1 - P)T^\diamond\) where \(P\) is the projector onto \(\ker T\) along \(B^\perp\) and \(T^\diamond\) is an arbitrary right inverse of \(T\).

For a regular dual problem \((S, G)\), the Green’s operator is given by \(T = G^\diamond(1 - P)\) where \(P\) is the projector onto \(S\) along \(\text{im} \ G\) and \(G^\diamond\) is an arbitrary left inverse of \(G\).
Proposition

For a regular boundary problem \((T, F)\), the Green’s operator is given by 
\[ G = (1 - P)T^\diamond \] where \(P\) is the projector onto \(\ker T\) along \(B^\perp\) and \(T^\diamond\) is an arbitrary right inverse of \(T\).

For a regular dual problem \((S, G)\), the Green’s operator is given by 
\[ T = G^\diamond (1 - P) \] where \(P\) is the projector onto \(S\) along \(\text{im} G\) and \(G^\diamond\) is an arbitrary left inverse of \(G\).

If \(\dim B < \infty\) or \(\dim S < \infty\) then:
Proposition

For a regular boundary problem \((T, \mathcal{F})\), the Green’s operator is given by \(G = (1 - P)T^{\diamond}\) where \(P\) is the projector onto \(\ker T\) along \(\mathcal{B}^\perp\) and \(T^{\diamond}\) is an arbitrary right inverse of \(T\).

For a regular dual problem \((S, G)\), the Green’s operator is given by \(T = G^{\diamond}(1 - P)\) where \(P\) is the projector onto \(S\) along \(\text{im} G\) and \(G^{\diamond}\) is an arbitrary left inverse of \(G\).

If \(\dim \mathcal{B} < \infty\) or \(\dim S < \infty\) then:

- \(\mathcal{B} = [\beta_1, \ldots, \beta_n]\) and \(\ker T = [u_1, \ldots, u_n]\):
Proposition

For a regular boundary problem \((T, F)\), the Green’s operator is given by 
\[ G = (1 - P)T^{\diamond} \]
where \(P\) is the projector onto \(\ker T\) along \(B^\perp\) and \(T^{\diamond}\) is an arbitrary right inverse of \(T\).

For a regular dual problem \((S, G)\), the Green’s operator is given by 
\[ T = G^{\diamond}(1 - P) \]
where \(P\) is the projector onto \(S\) along \(\text{im} G\) and \(G^{\diamond}\) is an arbitrary left inverse of \(G\).

If \(\dim B < \infty\) or \(\dim S < \infty\) then:

- \(B = [\beta_1, \ldots, \beta_n]\) and \(\ker T = [u_1, \ldots, u_n]\):
  - Regularity \(\iff\) Evaluation matrix \(\beta(u) = [\beta_i(u_j)] \in GL_n(K)\)
Proposition

For a regular boundary problem $(T, F)$, the Green’s operator is given by $G = (1 - P)T^\diamond$ where $P$ is the projector onto $\ker T$ along $B^\perp$ and $T^\diamond$ is an arbitrary right inverse of $T$.

For a regular dual problem $(S, G)$, the Green’s operator is given by $T = G^\diamond(1 - P)$ where $P$ is the projector onto $S$ along $\text{im} G$ and $G^\diamond$ is an arbitrary left inverse of $G$.

If $\dim B < \infty$ or $\dim S < \infty$ then:

- $B = [\beta_1, \ldots, \beta_n]$ and $\ker T = [u_1, \ldots, u_n]$:
  - Regularity $\Leftrightarrow$ Evaluation matrix $\beta(u) = [\beta_i(u_j)] \in \text{GL}_n(K)$
  - Projector $P = u \cdot \beta(u)^{-1} \cdot \beta$
Proposition

For a regular boundary problem \((T, \mathcal{F})\), the Green’s operator is given by \(G = (1 - P)T^\diamond\) where \(P\) is the projector onto \(\ker T\) along \(\mathcal{B}^\perp\) and \(T^\diamond\) is an arbitrary right inverse of \(T\).

For a regular dual problem \((S, G)\), the Green’s operator is given by \(T = G^\diamond (1 - P)\) where \(P\) is the projector onto \(S\) along \(\text{im } G\) and \(G^\diamond\) is an arbitrary left inverse of \(G\).

If \(\dim \mathcal{B} < \infty\) or \(\dim S < \infty\) then:

- \(\mathcal{B} = [\beta_1, \ldots, \beta_n]\) and \(\ker T = [u_1, \ldots, u_n]::\)
  - Regularity \(\iff\) Evaluation matrix \(\beta(u) = [\beta_i(u_j)] \in \text{GL}_n(K)\)
  - Projector \(P = u \cdot \beta(u)^{-1} \cdot \beta\)

- Analogous for dual problem:
Determination of Green’s Operators

Proposition

For a regular boundary problem \((T, \mathcal{F})\), the Green’s operator is given by \(G = (1 - P)T^\diamond\) where \(P\) is the projector onto \(\ker T\) along \(\mathcal{B}^\perp\) and \(T^\diamond\) is an arbitrary right inverse of \(T\).

For a regular dual problem \((S, G)\), the Green’s operator is given by \(T = G^\diamond(1 - P)\) where \(P\) is the projector onto \(S\) along \(\text{im} G\) and \(G^\diamond\) is an arbitrary left inverse of \(G\).

If \(\dim \mathcal{B} < \infty\) or \(\dim S < \infty\) then:

- \(\mathcal{B} = [\beta_1, \ldots, \beta_n]\) and \(\ker T = [u_1, \ldots, u_n]\):
  - Regularity \(\Leftrightarrow\) Evaluation matrix \(\beta(u) = [\beta_i(u_j)] \in \text{GL}_n(K)\)
  - Projector \(P = u \cdot \beta(u)^{-1} \cdot \beta\)

- Analogous for dual problem:
  - \(\text{im}^\perp G = [\beta_1, \ldots, \beta_n]\) and \(S = [u_1, \ldots, u_n]\)
Theorem

Let \((T, \mathcal{B}) \in \text{BnProb}^\ast\) and \(T = T_1T_2\) a factorization into epimorphisms. Then \((T, \mathcal{B}) = (T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2)\) is a factorization in \(\text{BnProb}^\ast\) iff

\[ \mathcal{B}_1 = H_2^\ast(\mathcal{B} \cap K_2^\perp) \quad \text{with} \quad K_2 := \ker T_2 \quad \text{and} \quad T_2H_2 = 1 \]

and \(\mathcal{B}_2 \leq \mathcal{B}\) is orthogonally closed such that \(\mathcal{B} = (\mathcal{B} \cap K_2^\perp) \hat{\oplus} \mathcal{B}_2\). In that case, \(G_1 = T_2G\).
Factorization of Boundary Problems

**Theorem**

Let \((T, \mathcal{B}) \in \text{BnProb}^*\) and \(T = T_1 T_2\) a factorization into epimorphisms. Then \((T, \mathcal{B}) = (T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2)\) is a factorization in \(\text{BnProb}^*\) iff

\[
\mathcal{B}_1 = H_2^*(\mathcal{B} \cap K_2^\perp) \quad \text{with} \quad K_2 := \ker T_2 \quad \text{and} \quad T_2 H_2 = 1
\]

and \(\mathcal{B}_2 \leq \mathcal{B}\) is orthogonally closed such that \(\mathcal{B} = (\mathcal{B} \cap K_2^\perp) \perp \mathcal{B}_2\).

In that case, \(G_1 = T_2 G\).

For fixed \(T = T_1 T_2\):
Factorization of Boundary Problems

**Theorem**

Let \((T, \mathcal{B}) \in \text{BnProb}^\ast\) and \(T = T_1 T_2\) a factorization into epimorphisms. Then \((T, \mathcal{B}) = (T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2)\) is a factorization in \(\text{BnProb}^\ast\) iff

\[
\mathcal{B}_1 = H^\ast_2(\mathcal{B} \cap K_2^\perp) \quad \text{with} \quad K_2 := \ker T_2 \quad \text{and} \quad T_2 H_2 = 1
\]

and \(\mathcal{B}_2 \leq \mathcal{B}\) is orthogonally closed such that \(\mathcal{B} = (\mathcal{B} \cap K_2^\perp) \hat{+} \mathcal{B}_2\).

In that case, \(G_1 = T_2 G\).

For fixed \(T = T_1 T_2\):

\[
\{ \mathcal{B}_2 \mid (T_2, \mathcal{B}_2) \in \text{BnProb}^\ast \} \quad \longleftrightarrow \quad \{ L_2 \mid K_2 \hat{+} L_2 = \ker T \}
\]
Factorization of Boundary Problems

Theorem

Let \((T, \mathcal{B}) \in \text{BnProb}^*\) and \(T = T_1 T_2\) a factorization into epimorphisms. Then \((T, \mathcal{B}) = (T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2)\) is a factorization in \(\text{BnProb}^*\) iff

\[\mathcal{B}_1 = H_2^*(\mathcal{B} \cap K_2^\perp)\]

with \(K_2 := \ker T_2\) and \(T_2 H_2 = 1\)

and \(\mathcal{B}_2 \leq \mathcal{B}\) is orthogonally closed such that \(\mathcal{B} = (\mathcal{B} \cap K_2^\perp) \dot{+} \mathcal{B}_2\).

In that case, \(G_1 = T_2 G\).

For fixed \(T = T_1 T_2\):

\[
\begin{align*}
\{ \mathcal{B}_2 \mid (T_2, \mathcal{B}_2) \in \text{BnProb}^* \} & \iff \{ L_2 \mid K_2 \dot{+} L_2 = \ker T \} \\
\mathcal{B}_2 & \mapsto \mathcal{B}_2 ^\perp \cap \ker T \\
\mathcal{B} \cap L_2^\perp & \mapsto L_2
\end{align*}
\]
Incarnations of Boundary Problems

- Generic boundary problem $(T, B)$, regularity $\iff$ unique solvability of:

  \[
  \text{Semi-Inhomogeneous Boundary Problem:} \\
  Tu = f \beta(u) = 0
  \]

  \[
  \text{Signal Operator = Semi-Inhomogeneous Green's Operator} \\
  f \to u
  \]

  \[
  \text{Semi-Homogeneous Boundary Problem:} \\
  Tu = 0 \beta(u) = B(\beta)
  \]

  \[
  \text{State Operator = Semi-Homogeneous Green's Operator} \\
  B \to u
  \]

  \[
  \text{Fully Inhomogeneous Boundary Problem:} \\
  Tu = f \beta(u) = B(\beta)
  \]

  \[
  \text{Full Operator = Fully Inhomogeneous Green's Operator} \\
  (f, B) \to u
  \]

  \[
  \text{Fully Homogeneous Boundary Problem:} \\
  Tu = 0 \beta(u) = 0
  \]

  \[
  \text{Trivial:} \\
  u = 0
  \]
Incarnations of Boundary Problems

Generic boundary problem \((T, B)\),
Incarnations of Boundary Problems

Generic boundary problem \((\mathcal{T}, \mathcal{B})\), regularity \(\iff\) unique solvability of: 

**Semi-Inhomogeneous Boundary Problem:**
\[
\mathcal{T} u = f, \quad \beta(u) = 0
\]

**Signal Operator** = Semi-Inhomogeneous Green's Operator \(G\):
\[
 f \mapsto u
\]

**Semi-Homogeneous Boundary Problem:**
\[
\mathcal{T} u = 0, \quad \beta(u) = \mathcal{B}(\beta)
\]

**State Operator** = Semi-Homogeneous Green's Operator \(H\):
\[
\mathcal{B} \mapsto u
\]

**Fully Inhomogeneous Boundary Problem:**
\[
\mathcal{T} u = f, \quad \beta(u) = \mathcal{B}(\beta)
\]

**Full Operator** = Fully Inhomogeneous Green's Operator \(F\):
\[
(f, \mathcal{B}) \mapsto u
\]

**Fully Homogeneous Boundary Problem:**
\[
\mathcal{T} u = 0, \quad \beta(u) = 0
\]

**Trivial:**
\[
u = 0
\]
Incarnations of Boundary Problems

Generic boundary problem \((T, B)\), regularity \(\iff\) unique solvability of:

[Semi-Inhomogeneous] Boundary Problem:

\[
\begin{align*}
Tu &= f \\
\beta(u) &= 0
\end{align*}
\]

Signal Operator = [Semi-Inhomogeneous] Green's Operator
\(G: f \mapsto u\)
Incarnations of Boundary Problems

Generic boundary problem \((T, B)\), regularity \(\Leftrightarrow\) unique solvability of:

**[Semi-Inhomogeneous] Boundary Problem:**

\[
\begin{align*}
Tu &= f \\
\beta(u) &= 0
\end{align*}
\]

Signal Operator = [Semi-Inhomogeneous] Green’s Operator
\(G: f \mapsto u\)

**Semi-Homogeneous Boundary Problem:**

\[
\begin{align*}
Tu &= 0 \\
\beta(u) &= B(\beta)
\end{align*}
\]

State Operator = Semi-Homogeneous Green’s Operator
\(H: B \mapsto u\)
Incarnations of Boundary Problems

Generic boundary problem \((T, B)\), regularity \(\iff\) unique solvability of:

[Semi-Inhomogeneous] Boundary Problem:

\[
egin{align*}
Tu &= f \\
\beta(u) &= 0
\end{align*}
\]

Signal Operator = [Semi-Inhomogeneous] Green's Operator
\[G: f \mapsto u\]

Semi-Homogeneous Boundary Problem:

\[
egin{align*}
Tu &= 0 \\
\beta(u) &= B(\beta)
\end{align*}
\]

State Operator = Semi-Homogeneous Green’s Operator
\[H: B \mapsto u\]

Fully Inhomogeneous Boundary Problem:

\[
egin{align*}
Tu &= f \\
\beta(u) &= B(\beta)
\end{align*}
\]

Full Operator = Fully Inhomogeneous Green’s Operator
\[F: (f, B) \mapsto u\]
Incarnations of Boundary Problems

Generic boundary problem \((T, B)\), regularity \(\iff\) unique solvability of:

[Semi-Inhomogeneous] Boundary Problem:
\[
\begin{align*}
Tu &= f \\
\beta(u) &= 0
\end{align*}
\]
Signal Operator = [Semi-Inhomogeneous] Green’s Operator
\(G: f \mapsto u\)

Semi-Homogeneous Boundary Problem:
\[
\begin{align*}
Tu &= 0 \\
\beta(u) &= B(\beta)
\end{align*}
\]
State Operator = Semi-Homogeneous Green’s Operator
\(H: B \mapsto u\)

Fully Inhomogeneous Boundary Problem:
\[
\begin{align*}
Tu &= f \\
\beta(u) &= B(\beta)
\end{align*}
\]
Full Operator = Fully Inhomogeneous Green’s Operator
\(F: (f, B) \mapsto u\)

Fully Homogeneous Boundary Problem:
\[
\begin{align*}
Tu &= 0 \\
\beta(u) &= 0
\end{align*}
\]
Trivial: \(u = 0\)
Assume $(T, B) \in \text{BnProb}^*$ given:
Assume \((T, \mathcal{B}) \in \text{BnProb}^*\) given:

- Boundary basis \((\beta_i \mid i \in I)\) such that \(\mathcal{B} = [\beta_i \mid i \in I]\)
  - Linear span + orthogonal closure
Assume \((T, \mathcal{B}) \in \text{BnProb}^*\) given:

- Boundary basis \((\beta_i \mid i \in I)\) such that \(\mathcal{B} = [\beta_i \mid i \in I]\)
  Linear span + orthogonal closure
- Trace map \(\text{trc}: \mathcal{F} \to \mathcal{B}^*\) sends \(f \in \mathcal{F}\) to \(\beta \mapsto \beta(f)\)
Assume \((T, \mathcal{B}) \in \text{BnProb}^*\) given:

- Boundary basis \((\beta_i \mid i \in I)\) such that \(\mathcal{B} = [\beta_i \mid i \in I]\)
  Linear span + orthogonal closure

- Trace map \(\text{trc}: \mathcal{F} \rightarrow \mathcal{B}^*\) sends \(f \in \mathcal{F}\) to \(\beta \mapsto \beta(f)\)

- Boundary data \(B \in \mathcal{B}' := \text{im}(\text{trc})\)
Assume \((T, \mathcal{B}) \in \text{BnProb}^*\) given:

- Boundary basis \((\beta_i \mid i \in I)\) such that \(\mathcal{B} = [\beta_i \mid i \in I]\)
  - Linear span + orthogonal closure
- Trace map \(\text{trc} : \mathcal{F} \rightarrow \mathcal{B}^*\) sends \(f \in \mathcal{F}\) to \(\beta \mapsto \beta(f)\)
- Boundary data \(B \in \mathcal{B}' := \text{im}(\text{trc})\)
- Boundary values \(\overline{B} := B(\beta_i)_{i \in I} \in K^I\)
Boundary Data and Boundary Values

Assume \((T, \mathcal{B}) \in \text{BnProb}^*\) given:

- Boundary basis \((\beta_i \mid i \in I)\) such that \(\mathcal{B} = [\beta_i \mid i \in I]\)
  Linear span + orthogonal closure
- Trace map \(\text{trc}: \mathcal{F} \rightarrow \mathcal{B}^*\) sends \(f \in \mathcal{F}\) to \(\beta \mapsto \beta(f)\)
- Boundary data \(B \in \mathcal{B}' := \text{im} (\text{trc})\)
- Boundary values \(\overline{B} := B(\beta_i)_{i \in I} \in K^I\)

**Boundary Data** \(B \in \mathcal{B}'\)
- basis-free

**Boundary Basis** \((\beta_i)_{i \in I}\)

**Boundary Values** \(\overline{B} \in K^I\)
- basis-dependent
Boundary Data and Boundary Values

Assume \((T, \mathcal{B}) \in \text{BnProb}^*\) given:

- Boundary basis \((\beta_i \mid i \in I)\) such that \(\mathcal{B} = [\beta_i \mid i \in I]\)
  - Linear span + orthogonal closure
- Trace map \(\text{trc} : \mathcal{F} \to \mathcal{B}^*\) sends \(f \in \mathcal{F}\) to \(\beta \mapsto \beta(f)\)
- Boundary data \(B \in \mathcal{B}' := \text{im}(\text{trc})\)
- Boundary values \(\overline{B} := \beta(\beta_i)_{i \in I} \in K^I\)

<table>
<thead>
<tr>
<th>Boundary Data</th>
<th>Boundary Basis ((\beta_i)_{i \in I})</th>
<th>Boundary Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B \in \mathcal{B}')</td>
<td>basis-free</td>
<td>(\overline{B} \in K^I)</td>
</tr>
</tbody>
</table>

**Lemma**

Let \(\mathcal{B} \leq \mathcal{F}^*\) be a boundary space with boundary basis \((\beta_i \mid i \in I)\). If for any \(B_1, B_2 \in \mathcal{B}'\) one has \(\overline{B_1} = \overline{B_2}\) then also \(B_1 = B_2\). In particular, the trace \(f^*\) of \(f \in \mathcal{F}\) depends only on \(\overline{f^*} = \beta_i(f)_{i \in I} \in K^I\).
Definition

An interpolator for $B$ is a section $B \hat{\cdot} : B' \to \mathcal{F}$ of $\text{trc} : \mathcal{F} \to B'$.
An **interpolator** for \( \mathcal{B} \) is a section \( \mathcal{B}^\dagger : \mathcal{B}' \to \mathcal{F} \) of \( \text{trc} : \mathcal{F} \to \mathcal{B}' \).

Relative to \( (\beta_i \mid i \in I) \), it is given by a map \( K^I \to \mathcal{F} \).
Definition

An **interpolator** for $\mathcal{B}$ is a section $\mathcal{B}^{\diamond} : \mathcal{B}' \rightarrow \mathcal{F}$ of $\text{trc} : \mathcal{F} \rightarrow \mathcal{B}'$.

Relative to $(\beta_i \mid i \in I)$, it is given by a map $K^I \rightarrow \mathcal{F}$.

Boundary values $\overline{\mathcal{B}} \in K^I \leadsto$ boundary data $\mathcal{B} \in \mathcal{B}'$ via $\mathcal{B}(\beta) := \beta(\mathcal{B}^{\diamond} \overline{\mathcal{B}})$.
Definition

An interpolator for $\mathcal{B}$ is a section $\mathcal{B} \dagger: \mathcal{B}' \to \mathcal{F}$ of $\text{trc}: \mathcal{F} \to \mathcal{B}'$.

Relative to $(\beta_i \mid i \in I)$, it is given by a map $K^I \to \mathcal{F}$.

Boundary values $\overline{B} \in K^I \leadsto$ boundary data $B \in \mathcal{B}'$ via $B(\beta) := \beta(\mathcal{B} \dagger \overline{B})$.

Theorem

Let $(T, \mathcal{B}) \in \text{BnProb}^*$ be given. Then $G = (1 - P) T \dagger$ and $H = P \mathcal{B} \dagger$, hence $F = (1 - P) T \dagger \oplus P \mathcal{B} \dagger$, where $P$ projects onto $\ker T$ along $\mathcal{B} \perp$. 
Interpolator and Green’s Operators

Definition

An interpolator for $\mathcal{B}$ is a section $\mathcal{B}^{\diamond} : \mathcal{B}' \to \mathcal{F}$ of $\text{trc} : \mathcal{F} \to \mathcal{B}'$. Relative to $(\beta_i \mid i \in I)$, it is given by a map $K^I \to \mathcal{F}$. Boundary values $\overline{B} \in K^I \leadsto$ boundary data $B \in \mathcal{B}'$ via $B(\beta) := \beta(\mathcal{B}^{\diamond} \cdot \overline{B})$.

Theorem

Let $(T, \mathcal{B}) \in \text{BnProb}^*$ be given. Then $G = (1 - P) T^{\diamond}$ and $H = P \mathcal{B}^{\diamond}$, hence $F = (1 - P) T^{\diamond} \oplus P \mathcal{B}^{\diamond}$, where $P$ projects onto $\ker T$ along $\mathcal{B}^\perp$.

Usually more realistic to compute $P$ from $H$:
Interpolator and Green’s Operators

**Definition**

An **interpolator** for $\mathcal{B}$ is a section $\mathcal{B}^\diamondsuit : \mathcal{B}' \to \mathcal{F}$ of $\text{trc} : \mathcal{F} \to \mathcal{B}'$.

Relative to $(\beta_i \mid i \in I)$, it is given by a map $K^I \to \mathcal{F}$.

Boundary values $\overline{\beta} \in K^I \rightsquigarrow$ boundary data $\beta \in \mathcal{B}'$ via $\beta(\beta) := \beta(\mathcal{B}^\diamondsuit \overline{\beta})$.

**Theorem**

Let $(T, \mathcal{B}) \in \text{BnProb}^*$ be given. Then $G = (1 - P) T^\diamondsuit$ and $H = P \mathcal{B}^\diamondsuit$, hence $F = (1 - P) T^\diamondsuit \oplus P \mathcal{B}^\diamondsuit$, where $P$ projects onto $\ker T$ along $\mathcal{B}^\perp$.

Usually more realistic to compute $P$ from $H$:

**Proposition**

Let $(T, \mathcal{B}) \in \text{BnProb}^*$ be given. Then $\text{trc}|_{\ker T}$ is bijective with state operator $H$ as inverse, and the kernel projector is $P = H \circ \text{trc}$.
LODE Example: Two-Point Boundary Problem

Given a forcing function $f(x) \in C^\infty[a,b]$ and boundary data $(\rho, \sigma) \in \mathbb{R}^2$, find a solution $u(x) \in C^\infty[a,b]$ such that

\[ u'' = f, \quad u(a) + u(b) = \rho, \quad u'(b) - u(b) = \sigma. \]

Key elements:
- Function space $F = C^\infty[a,b]$
- Boundary space $B = [L + R, RD - R]$, $I = \{1, 2\}$
- Boundary basis $\{\beta_i | i \in I\}$ with $\beta_1 = L + R, \beta_2 = RD - R$
- Boundary data $B = (L + R \mapsto \rho, RD - R \mapsto \sigma) \in B'$
- Boundary values $B = (\rho, \sigma) \in \mathbb{R}^I$
Given a **forcing function** \( f(x) \in C^\infty[a, b] \) and **boundary data** \((\rho, \sigma) \in \mathbb{R}^2\), find **solution** \( u(x) \in C^\infty[a, b] \) such that

\[
\begin{align*}
u'' &= f, \\
u(a) + u(b) &= \rho, \\
u'(b) - u(b) &= \sigma.
\end{align*}
\]
LODE Example: Two-Point Boundary Problem

Given a **forcing function** $f(x) \in C^\infty[a, b]$ and **boundary data** $(\rho, \sigma) \in \mathbb{R}^2$,
find **solution** $u(x) \in C^\infty[a, b]$ such that

\[
\begin{align*}
  u'' &= f, \\
  u(a) + u(b) &= \rho, \\
  u'(b) - u(b) &= \sigma.
\end{align*}
\]

Key elements:
- Function space $\mathcal{F} = C^\infty[a, b]$
LODE Example: Two-Point Boundary Problem

Given a forcing function $f(x) \in C^\infty[a, b]$ and boundary data $(\rho, \sigma) \in \mathbb{R}^2$, find solution $u(x) \in C^\infty[a, b]$ such that

\[
\begin{align*}
    u'' &= f, \\
    u(a) + u(b) &= \rho, \\
    u'(b) - u(b) &= \sigma.
\end{align*}
\]

Key elements:

- Function space $\mathcal{F} = C^\infty[a, b]$
- Boundary space $\mathcal{B} = [L + R, RD - R], \ I = \{1, 2\}$
LODE Example: Two-Point Boundary Problem

Given a **forcing function** \( f(x) \in C^\infty[a, b] \) and **boundary data** \((\rho, \sigma) \in \mathbb{R}^2\), find **solution** \( u(x) \in C^\infty[a, b] \) such that

\[
\begin{align*}
    u'' &= f, \\
    u(a) + u(b) &= \rho, \\
    u'(b) - u(b) &= \sigma.
\end{align*}
\]

**Key elements:**

- **Function space** \( \mathcal{F} = C^\infty[a, b] \)
- **Boundary space** \( \mathcal{B} = [L + R, RD - R], \ I = \{1, 2\} \)
- **Boundary basis** \( (\beta_i \mid i \in I) \) with \( \beta_1 = L + R, \beta_2 = RD - R \)
LODE Example: Two-Point Boundary Problem

Given a forcing function \( f(x) \in C^\infty[a, b] \) and boundary data \( (\rho, \sigma) \in \mathbb{R}^2 \), find solution \( u(x) \in C^\infty[a, b] \) such that

\[
\begin{align*}
  u'' &= f, \\
  u(a) + u(b) &= \rho, \\
  u'(b) - u(b) &= \sigma.
\end{align*}
\]

Key elements:

- Function space \( \mathcal{F} = C^\infty[a, b] \)
- Boundary space \( \mathcal{B} = [L + R, RD - R], \ I = \{1, 2\} \)
- Boundary basis \( (\beta_i \mid i \in I) \) with \( \beta_1 = L + R, \beta_2 = RD - R \)
- Boundary data \( B = (L + R \mapsto \rho, RD - R \mapsto \sigma) \in \mathcal{B}' \)
LODE Example: Two-Point Boundary Problem

Given a **forcing function** $f(x) \in C^\infty[a, b]$ and **boundary data** $(\rho, \sigma) \in \mathbb{R}^2$,
find **solution** $u(x) \in C^\infty[a, b]$ such that

\[
\begin{align*}
    u'' &= f, \\
    u(a) + u(b) &= \rho, \\
    u'(b) - u(b) &= \sigma.
\end{align*}
\]

**Key elements:**

- **Function space** $\mathcal{F} = C^\infty[a, b]$
- **Boundary space** $\mathcal{B} = [L + R, RD - R]$, $I = \{1, 2\}$
- **Boundary basis** $(\beta_i \mid i \in I)$ with $\beta_1 = L + R, \beta_2 = RD - R$
- **Boundary data** $B = (L + R \mapsto \rho, RD - R \mapsto \sigma) \in \mathcal{B}'$
- **Boundary values** $\overline{B} = (\rho, \sigma) \in \mathbb{R}^I$
LODE Example: Two-Point Boundary Problem

Given a forcing function $f(x) \in C^\infty[a, b]$ and boundary data $(\rho, \sigma) \in \mathbb{R}^2$, find solution $u(x) \in C^\infty[a, b]$ such that

\[
\begin{align*}
    u'' &= f, \\
    u(a) + u'(b) &= \rho + \sigma, \\
    u(b) - u'(b) &= -\sigma.
\end{align*}
\]

Key elements:
- Function space $\mathcal{F} = C^\infty[a, b]$
- Boundary space $\mathcal{B} = [L + R, RD - R]$, $I = \{1, 2\}$
- Boundary basis $(\beta_i \mid i \in I)$ with $\beta_1 = L + R, \beta_2 = RD - R$
- Boundary data $B = (L + R \mapsto \rho, RD - R \mapsto \sigma) \in \mathcal{B}'$
- Boundary values $\overline{B} = (\rho, \sigma) \in \mathbb{R}^I$
Given a forcing function \( f(x) \in C^\infty[a, b] \) and boundary data \((\rho, \sigma) \in \mathbb{R}^2\), find solution \( u(x) \in C^\infty[a, b] \) such that

\[
\begin{align*}
  u'' &= f, \\
  u(a) + u'(b) &= \rho + \sigma, \\
  u(b) - u'(b) &= -\sigma.
\end{align*}
\]

Key elements:

- Function space \( \mathcal{F} = C^\infty[a, b] \)
- Boundary space \( \mathcal{B} = [L + RD, R - RD], \ I = \{1, 2\} \)
- New basis \((\gamma_i \mid i \in I)\) with \( \gamma_1 = L + RD, \gamma_2 = R - RD \)
- Boundary data \( \mathcal{B} = (L + R \mapsto \rho, RD - R \mapsto \sigma) \in \mathcal{B}' \)
- Boundary values \( \mathcal{B} = (\rho, \sigma) \in \mathbb{R}^I \)
Given a forcing function $f(x) \in C^\infty[a, b]$ and boundary data $(\rho, \sigma) \in \mathbb{R}^2$, find solution $u(x) \in C^\infty[a, b]$ such that

\[
\begin{align*}
\frac{d^2 u}{dx^2} &= f, \\
u(a) + u'(b) &= \rho + \sigma, \\
u(b) - u'(b) &= -\sigma.
\end{align*}
\]

Key elements:

- Function space $\mathcal{F} = C^\infty[a, b]$
- Boundary space $\mathcal{B} = [L + RD, R - RD], \ I = \{1, 2\}$
- **New** basis $(\gamma_i \mid i \in I)$ with $\gamma_1 = L + RD, \gamma_2 = R - RD$
- Boundary data $B = (L + RD \mapsto \rho + \sigma, R - RD \mapsto -\sigma) \in \mathcal{B}'$
- Boundary values $\overline{B} = (\rho, \sigma) \in \mathbb{R}^I$
Given a **forcing function** \( f(x) \in C^\infty[a, b] \) and boundary data \((\rho, \sigma) \in \mathbb{R}^2\), find a **solution** \( u(x) \in C^\infty[a, b] \) such that

\[
\begin{align*}
\frac{d^2}{dx^2} u &= f, \\
u(a) + u'(b) &= \rho + \sigma, \quad u(b) - u'(b) = -\sigma.
\end{align*}
\]

Key elements:

- **Function space** \( \mathcal{F} = C^\infty[a, b] \)
- **Boundary space** \( \mathcal{B} = [L + RD, R - RD] \), \( I = \{1, 2\} \)
- **New** basis \( (\gamma_i \mid i \in I) \) with \( \gamma_1 = L + RD, \gamma_2 = R - RD \)
- **Boundary data** \( B = (L + RD \mapsto \rho + \sigma, R - RD \mapsto -\sigma) \in \mathcal{B}' \)
- **New** values \( \bar{C} = (\bar{\rho}, \bar{\sigma}) = (\rho + \sigma, -\sigma) \in \mathbb{R}^I \)
LODE Example: Two-Point Boundary Problem

Given a **forcing function** \( f(x) \in C^\infty[a, b] \) and **boundary data** \((\rho, \sigma) \in \mathbb{R}^2\),
find **solution** \( u(x) \in C^\infty[a, b] \) such that

\[
\begin{align*}
u'' &= f, \\
u(a) + u(b) &= \rho, \\
u'(b) - u(b) &= \sigma.
\end{align*}
\]

**Key elements:**
- Function space \( \mathcal{F} = C^\infty[a, b] \)
- Boundary space \( \mathcal{B} = [L + R, RD - R], \ I = \{1, 2\} \)
- Boundary basis \( (\beta_i \mid i \in I) \) with \( \beta_1 = L + R, \beta_2 = RD - R \)
- Boundary data \( B = (L + R \mapsto \rho, RD - R \mapsto \sigma) \in \mathcal{B}' \)
- Boundary values \( \overline{B} = (\rho, \sigma) \in \mathbb{R}^I \)
LPDE Example: Cauchy Problem

Given a forcing function \( f(t,x,y) \in C^\omega(\mathbb{R}^3) \) and Cauchy data \( f_1(x,y), f_2(x,y) \in C^\omega(\mathbb{R}^2) \), find solution \( u(t,x,y) \in C^\omega(\mathbb{R}^3) \) such that

\[
\begin{align*}
    u_{tt} - 4u_{tx} + 4u_{xx} - 9u_{yy} &= f \\
    u(0,x,y) &= f_1(x,y) \\
    u_t(0,x,y) &= f_2(x,y)
\end{align*}
\]

Key elements:
- Function space \( F = C^\omega(\mathbb{R}^3) \)
- Boundary space \( B = \{E_0,x,y,E_0,x,y D_t | (x,y) \in \mathbb{R}^2\} \), \( I = \mathbb{R}^2 \sqcup \mathbb{R}^2 \)
- Boundary basis \( (\beta_i | i \in I) \) with \( \beta(x,y) \), \( \beta(x,y) \)
- Boundary data \( B = (E_0,x,y \mapsto f_1(x,y), E_0,x,y D_t \mapsto f_2(x,y)) \in B' \)
- Boundary values \( B = (f_1,f_2) \in \mathbb{R}^I \)

Markus Rosenkranz
Differential Algebra for Boundary Problems
Given a **forcing function** \( f(t, x, y) \in C^\omega(\mathbb{R}^3) \) and **Cauchy data** \( f_1(x, y), f_2(x, y) \in C^\omega(\mathbb{R}^2) \), find **solution** \( u(t, x, y) \in C^\omega(\mathbb{R}^3) \) such that

\[
\begin{align*}
  u_{tt} - 4u_{tx} + 4u_{xx} - 9u_{yy} &= f \\
  u(0, x, y) &= f_1(x, y), \quad u_t(0, x, y) = f_2(x, y)
\end{align*}
\]
Given a **forcing function** \( f(t, x, y) \in C^\omega(\mathbb{R}^3) \) and **Cauchy data** \( f_1(x, y), f_2(x, y) \in C^\omega(\mathbb{R}^2) \), find **solution** \( u(t, x, y) \in C^\omega(\mathbb{R}^3) \) such that

\[
\begin{align*}
  u_{tt} - 4u_{tx} + 4u_{xx} - 9u_{yy} &= f \\
  u(0, x, y) &= f_1(x, y), \quad u_t(0, x, y) = f_2(x, y)
\end{align*}
\]

Key elements:
- Function space \( \mathcal{F} = C^\omega(\mathbb{R}^3) \)
Given a forcing function \( f(t, x, y) \in C^\omega(\mathbb{R}^3) \) and Cauchy data \( f_1(x, y), f_2(x, y) \in C^\omega(\mathbb{R}^2) \), find solution \( u(t, x, y) \in C^\omega(\mathbb{R}^3) \) such that

\[
\begin{align*}
    u_{tt} - 4u_{tx} + 4u_{xx} - 9u_{yy} &= f \\
    u(0, x, y) &= f_1(x, y), \\
    u_t(0, x, y) &= f_2(x, y)
\end{align*}
\]

Key elements:
- Function space \( \mathcal{F} = C^\omega(\mathbb{R}^3) \)
- Boundary space \( \mathcal{B} = [E_{0,x,y}, E_{0,x,y}D_t | (x, y) \in \mathbb{R}^2], \quad I = \mathbb{R}^2 \sqcup \mathbb{R}^2 \)
Given a **forcing function** \( f(t, x, y) \in C^\omega(\mathbb{R}^3) \) and **Cauchy data** \( f_1(x, y), f_2(x, y) \in C^\omega(\mathbb{R}^2) \), find **solution** \( u(t, x, y) \in C^\omega(\mathbb{R}^3) \) such that

\[
\begin{align*}
  u_{tt} - 4u_{tx} + 4u_{xx} - 9u_{yy} &= f \\
  u(0, x, y) &= f_1(x, y), \quad u_t(0, x, y) = f_2(x, y)
\end{align*}
\]

Key elements:

- **Function space** \( \mathcal{F} = C^\omega(\mathbb{R}^3) \)
- **Boundary space** \( \mathcal{B} = \left[ E_{0,x,y}, E_{0,x,y}D_t \mid (x, y) \in \mathbb{R}^2 \right], \quad I = \mathbb{R}^2 \sqcup \mathbb{R}^2 \)
- **Boundary basis** \( (\beta_i \mid i \in I) \) with \( \beta_{(x,y),1} = E_{0,x,y}, \beta_{(x,y),2} = E_{0,x,y}D_t \)
Given a **forcing function** \( f(t, x, y) \in C^\omega(\mathbb{R}^3) \) and **Cauchy data** \( f_1(x, y), f_2(x, y) \in C^\omega(\mathbb{R}^2) \), find **solution** \( u(t, x, y) \in C^\omega(\mathbb{R}^3) \) such that

\[
\begin{align*}
  u_{tt} - 4u_{tx} + 4u_{xx} - 9u_{yy} &= f \\
  u(0, x, y) &= f_1(x, y), \quad u_t(0, x, y) = f_2(x, y)
\end{align*}
\]

**Key elements:**

- Function space \( \mathcal{F} = C^\omega(\mathbb{R}^3) \)
- Boundary space \( \mathcal{B} = \{E_{0,x,y}, E_{0,x,y}D_t \mid (x, y) \in \mathbb{R}^2\} \), \( I = \mathbb{R}^2 \sqcup \mathbb{R}^2 \)
- Boundary basis \((\beta_i \mid i \in I)\) with \( \beta(x,y),_{1} = E_{0,x,y}, \beta(x,y),_{2} = E_{0,x,y}D_t \)
- Boundary data \( \mathcal{B} = (E_{0,x,y} \mapsto f_1(x, y), E_{0,x,y}D_t \mapsto f_2(x, y)) \in \mathcal{B}' \)
LPDE Example: Cauchy Problem

Given a **forcing function** \( f(t, x, y) \in C^\omega(\mathbb{R}^3) \) and **Cauchy data** \( f_1(x, y), f_2(x, y) \in C^\omega(\mathbb{R}^2) \), find **solution** \( u(t, x, y) \in C^\omega(\mathbb{R}^3) \) such that

\[
\begin{align*}
  u_{tt} - 4u_{tx} + 4u_{xx} - 9u_{yy} &= f \\
  u(0, x, y) &= f_1(x, y), \quad u_t(0, x, y) = f_2(x, y)
\end{align*}
\]

**Key elements:**

- **Function space** \( \mathcal{F} = C^\omega(\mathbb{R}^3) \)
- **Boundary space** \( \mathcal{B} = \{ E_{0,x,y}, E_{0,x,y}D_t \mid (x, y) \in \mathbb{R}^2 \} \), \( I = \mathbb{R}^2 \sqcup \mathbb{R}^2 \)
- **Boundary basis** \( (\beta_i \mid i \in I) \) with \( \beta_{(x,y),1} = E_{0,x,y}, \beta_{(x,y),2} = E_{0,x,y}D_t \)
- **Boundary data** \( B = (E_{0,x,y} \mapsto f_1(x, y), E_{0,x,y}D_t \mapsto f_2(x, y)) \in \mathcal{B}' \)
- **Boundary values** \( \overline{B} = (f_1, f_2) \in \mathbb{R}^I \)
LPDE Example: Cauchy Problem

Given a **forcing function** \( f(t, x, y) \in C^\omega(\mathbb{R}^3) \) and **Cauchy data** \( f_1(x, y), f_2(x, y) \in C^\omega(\mathbb{R}^2) \), find **solution** \( u(t, x, y) \in C^\omega(\mathbb{R}^3) \) such that

\[
\begin{align*}
    u_{tt} - 4u_{tx} + 4u_{xx} - 9u_{yy} &= f \\
    u(0, 0, y) &= f_1(0, y), \quad u_x(0, x, y) = f_{1x}(x, y), \quad u_t(0, x, y) = f_2(x, y)
\end{align*}
\]

**Key elements:**

- Function space \( \mathcal{F} = C^\omega(\mathbb{R}^3) \)
- Boundary space \( \mathcal{B} = [E_{0,x,y}, E_{0,x,y}D_t \mid (x, y) \in \mathbb{R}^2], \ I = \mathbb{R}^2 \sqcup \mathbb{R}^2 \)
- Boundary basis \( (\beta_i \mid i \in I) \) with \( \beta_{(x,y),1} = E_{0,x,y}, \beta_{(x,y),2} = E_{0,x,y}D_t \)
- Boundary data \( \mathcal{B} = (E_{0,x,y} \mapsto f_1(x, y), E_{0,x,y}D_t \mapsto f_2(x, y)) \in \mathcal{B}' \)
- Boundary values \( \mathcal{B} = (f_1, f_2) \in \mathbb{R}^I \)
Given a forcing function $f(t, x, y) \in C^\omega(\mathbb{R}^3)$ and Cauchy data $f_1(x, y), f_2(x, y) \in C^\omega(\mathbb{R}^2)$, find solution $u(t, x, y) \in C^\omega(\mathbb{R}^3)$ such that

\[
    u(t, x, y) = f_1(0, y), \quad u_x(0, x, y) = f_{1x}(x, y), \quad u_t(0, x, y) = f_2(x, y)
\]

Key elements:

- Function space $\mathcal{F} = C^\omega(\mathbb{R}^3)$
- $\mathcal{B} = [E_{0,0,y}, E_{0,x,y}D_x, E_{0,x,y}D_t | (x, y) \in \mathbb{R}^2], \quad I = \mathbb{R} \sqcup \mathbb{R}^2 \sqcup \mathbb{R}^2$
- Boundary basis $(\beta_i | i \in I)$ with $\beta_{(x,y),1} = E_{0,x,y}, \beta_{(x,y),2} = E_{0,x,y}D_t$
- Boundary data $B = (E_{0,x,y} \mapsto f_1(x, y), \quad E_{0,x,y}D_t \mapsto f_2(x, y)) \in \mathcal{B}'$
- Boundary values $\overline{B} = (f_1, f_2) \in \mathbb{R}^I$
Given a forcing function \( f(t, x, y) \in C^\omega(\mathbb{R}^3) \) and Cauchy data \( f_1(x, y), f_2(x, y) \in C^\omega(\mathbb{R}^2) \), find solution \( u(t, x, y) \in C^\omega(\mathbb{R}^3) \) such that

\[
\begin{align*}
  &u_{tt} - 4u_{tx} + 4u_{xx} - 9u_{yy} = f \\
  &u(0, 0, y) = f_1(0, y), \quad u_x(0, x, y) = f_{1x}(x, y), \quad u_t(0, x, y) = f_2(x, y)
\end{align*}
\]

Key elements:

- Function space \( \mathcal{F} = C^\omega(\mathbb{R}^3) \)
- \( \mathcal{B} = \left[ E_{0,0,y}, E_{0,x,y}D_x, E_{0,x,y}D_t \mid (x, y) \in \mathbb{R}^2 \right], \quad I = \mathbb{R} \sqcup \mathbb{R}^2 \sqcup \mathbb{R}^2 \)
- **New** basis \( \gamma_{y,1} = E_{0,0,y}, \quad \gamma(x,y),2 = E_{0,x,y}D_x, \quad \gamma(x,y),3 = E_{0,x,y}D_t \)
- Boundary data \( B = (E_{0,x,y} \mapsto f_1(x, y), \quad E_{0,x,y}D_t \mapsto f_2(x, y)) \in \mathcal{B}' \)
- Boundary values \( \overline{B} = (f_1, f_2) \in \mathbb{R}^I \)
Given a **forcing function** \( f(t, x, y) \in C^\omega(\mathbb{R}^3) \) and **Cauchy data** \( f_1(x, y), f_2(x, y) \in C^\omega(\mathbb{R}^2) \), find **solution** \( u(t, x, y) \in C^\omega(\mathbb{R}^3) \) such that

\[
\begin{align*}
    u_{tt} - 4u_{tx} + 4u_{xx} - 9u_{yy} &= f \\
    u(0, 0, y) &= f_1(0, y), \quad u_x(0, x, y) = f_1x(x, y), \quad u_t(0, x, y) = f_2(x, y)
\end{align*}
\]

**Key elements:**

- **Function space** \( F = C^\omega(\mathbb{R}^3) \)
- **Basis** \( \mathcal{B} = [E_{0,0,y}, E_{0,x,y}D_x, E_{0,x,y}D_t | (x, y) \in \mathbb{R}^2], \quad I = \mathbb{R} \cup \mathbb{R}^2 \cup \mathbb{R}^2 \)
- **New basis** \( \gamma_{y,1} = E_{0,0,y}, \quad \gamma(x,y),2 = E_{0,x,y}D_x, \quad \gamma(x,y),3 = E_{0,x,y}D_t \)
- **Boundary data** \( \mathcal{B} = (E_{0,0,y} \mapsto f_1(0, y), E_{0,x,y}D_x \mapsto f_1x(x, y), E_{0,x,y}D_t \mapsto f_2(x, y)) \in \mathcal{B}' \)
- **Boundary values** \( \overline{\mathcal{B}} = (f_1, f_2) \in \mathbb{R}^I \)
Given a **forcing function** \( f(t, x, y) \in C^\omega(\mathbb{R}^3) \) and **Cauchy data** \( f_1(x, y), f_2(x, y) \in C^\omega(\mathbb{R}^2) \), find **solution** \( u(t, x, y) \in C^\omega(\mathbb{R}^3) \) such that

\[
\begin{align*}
    u_{tt} - 4u_{tx} + 4u_{xx} - 9u_{yy} &= f \\
    u(0, 0, y) &= f_1(0, y), \\
    u_x(0, x, y) &= f_{1x}(x, y), \\
    u_t(0, x, y) &= f_2(x, y)
\end{align*}
\]

**Key elements:**

- **Function space** \( \mathcal{F} = C^\omega(\mathbb{R}^3) \)
- **Boundary data** \( \mathcal{B} = \left[ E_{0,0,y}, E_{0,x,y}D_x, E_{0,x,y}D_t \mid (x, y) \in \mathbb{R}^2 \right], \quad I = \mathbb{R} \sqcup \mathbb{R}^2 \sqcup \mathbb{R}^2 \)
- **New** basis \( \gamma_{y,1} = E_{0,0,y}, \quad \gamma(x,y),2 = E_{0,x,y}D_x, \quad \gamma(x,y),3 = E_{0,x,y}D_t \)
- **Boundary data** \( \mathcal{B} = (E_{0,0,y} \mapsto f_1(0, y), E_{0,x,y}D_x \mapsto f_{1x}(x, y), E_{0,x,y}D_t \mapsto f_2(x, y)) \in \mathcal{B}' \)
- **New** values \( \overline{C} = (g_1, g_2, g_3) = (f_1(0, y), f_{1x}(x, y), f_2(x, y)) \in \mathbb{R}^I \)
LPDE Example: Cauchy Problem

Given a forcing function \( f(t, x, y) \in C^\omega(\mathbb{R}^3) \) and Cauchy data \( f_1(x, y), f_2(x, y) \in C^\omega(\mathbb{R}^2) \), find solution \( u(t, x, y) \in C^\omega(\mathbb{R}^3) \) such that

\[
\begin{align*}
    u_{tt} - 4u_{tx} + 4u_{xx} - 9u_{yy} &= f \\
    u(0, x, y) &= f_1(x, y), \\n    u_t(0, x, y) &= f_2(x, y)
\end{align*}
\]

Key elements:

- Function space \( \mathcal{F} = C^\omega(\mathbb{R}^3) \)
- Boundary space \( \mathcal{B} = [E_{0,x,y}, E_{0,x,y}D_t \mid (x, y) \in \mathbb{R}^2], \ I = \mathbb{R}^2 \sqcup \mathbb{R}^2 \)
- Boundary basis \( (\beta_i \mid i \in I) \) with \( \beta(x,y),1 = E_{0,x,y}, \beta(x,y),2 = E_{0,x,y}D_t \)
- Boundary data \( B = (E_{0,x,y} \mapsto f_1(x, y), E_{0,x,y}D_t \mapsto f_2(x, y)) \in \mathcal{B}' \)
- Boundary values \( \overline{B} = (f_1, f_2) \in \mathbb{R}^I \)
Outline

1 Abstract Boundary Problems

2 Ordinary Integro-Differential Operators

3 Partial Integro-Differential Operators

4 Conclusion
Integro-Differential Algebras

**Definition**

Let \((\mathcal{F}, \partial)\) be a (noncommutative) differential algebra over a field \(K\). A \(K\)-linear operation \(\int : \mathcal{F} \rightarrow \mathcal{F}\) is called an integral operator for \(\partial\) if \(\partial \circ \int = 1_\mathcal{F}\) and the differential Rota-Baxter axiom

\[(\int f')(\int g') + \int (fg)' = (\int f')g + f(\int g')\]

is satisfied. Then \((\mathcal{F}, \partial, \int)\) is an integro-differential algebra.
Integro-Differential Algebras

Definition

Let \((\mathcal{F}, \partial)\) be a (noncommutative) differential algebra over a field \(K\). A \(K\)-linear operation \(\int : \mathcal{F} \rightarrow \mathcal{F}\) is called an integral operator for \(\partial\) if \(\partial \circ \int = 1_{\mathcal{F}}\) and the differential Rota-Baxter axiom
\[
(\int f')(\int g') + \int (fg)' = (\int f')g + f(\int g')
\]
is satisfied. Then \((\mathcal{F}, \partial, \int)\) is an integro-differential algebra.

Examples of integro-differential algebras:

- \(\mathcal{F} = C^\infty(\mathbb{R}^n), \partial u = u_{x_i}, \int u = \int_0^x u(\xi) d\xi\),

Markus Rosenkranz
Differential Algebra for Boundary Problems
Let \((\mathcal{F}, \partial)\) be a (noncommutative) differential algebra over a field \(K\). A \(K\)-linear operation \(\int : \mathcal{F} \to \mathcal{F}\) is called an integral operator for \(\partial\) if \(\partial \circ \int = 1_{\mathcal{F}}\) and the differential Rota-Baxter axiom
\[
(\int f')(\int g') + \int (fg)' = (\int f')g + f(\int g')
\]
is satisfied. Then \((\mathcal{F}, \partial, \int)\) is an integro-differential algebra.

Examples of integro-differential algebras:

- \(\mathcal{F} = C^\infty(\mathbb{R}^n), \partial u = u_{x_i}, \int u = \int_0^{x_i} u(\xi) \, d\xi\), partial for \(n > 1\)
Integro-Differential Algebras

Definition

Let \((\mathcal{F}, \partial)\) be a (noncommutative) differential algebra over a field \(K\). A \(K\)-linear operation \(\int : \mathcal{F} \to \mathcal{F}\) is called an integral operator for \(\partial\) if \(\partial \circ \int = 1_{\mathcal{F}}\) and the differential Rota-Baxter axiom

\[
(\int f') (\int g') + \int (fg)' = (\int f') g + f (\int g')
\]

is satisfied. Then \((\mathcal{F}, \partial, \int)\) is an integro-differential algebra.

Examples of integro-differential algebras:

- \(\mathcal{F} = C^\infty(\mathbb{R}^n), \quad \partial u = u_{x_i}, \quad \int u = \int_0^{x_i} u(\xi) \, d\xi\), partial for \(n > 1\)
- \(\mathcal{F} = C^\omega(D), \quad \partial u = u', \quad \int u = \int_0^z u(z) \, dz\)
**Definition**

Let \((\mathcal{F}, \partial)\) be a (noncommutative) differential algebra over a field \(K\). A \(K\)-linear operation \(\int : \mathcal{F} \to \mathcal{F}\) is called an **integral operator** for \(\partial\) if \(\partial \circ \int = 1_{\mathcal{F}}\) and the **differential Rota-Baxter axiom**

\[
(\int f')(\int g') + \int (fg)' = (\int f')g + f(\int g')
\]

is satisfied. Then \((\mathcal{F}, \partial, \int)\) is an **integro-differential algebra**.

Examples of integro-differential algebras:

- \(\mathcal{F} = C^\infty(\mathbb{R}^n), \partial u = u_{x_i}, \int u = \int_0^{x_i} u(\xi) \, d\xi\), partial for \(n > 1\)
- \(\mathcal{F} = C^\omega(D), \partial u = u', \int u = \int_0^z u(z) \, dz\)
- Holonomic functions \(\subset K[[x]]\)
Integro-Differential Algebras

**Definition**

Let \((\mathcal{F}, \partial)\) be a (noncommutative) differential algebra over a field \(K\). A \(K\)-linear operation \(\int : \mathcal{F} \to \mathcal{F}\) is called an integral operator for \(\partial\) if \(\partial \circ \int = 1_{\mathcal{F}}\) and the differential Rota-Baxter axiom

\[
(\int f')(\int g') + \int (fg)' = (\int f')g + f(\int g')
\]

is satisfied. Then \((\mathcal{F}, \partial, \int)\) is an integro-differential algebra.

Examples of integro-differential algebras:

- \(\mathcal{F} = C^\infty(\mathbb{R}^n), \partial u = u_{x_i}, \int u = \int_0^{x_i} u(\xi) \, d\xi,\) partial for \(n > 1\)
- \(\mathcal{F} = C^\omega(D), \partial u = u', \int u = \int_0^z u(z) \, dz\)
- Holonomic functions \(\subset K[[x]]\)
- Matrix rings \((\mathcal{F}^{n \times n}, \partial, \int)\)
Definition

Let \((\mathcal{F}, \partial)\) be a (noncommutative) differential algebra over a field \(K\). A \(K\)-linear operation \(\int: \mathcal{F} \rightarrow \mathcal{F}\) is called an integral operator for \(\partial\) if \(\partial \circ \int = 1_{\mathcal{F}}\) and the differential Rota-Baxter axiom

\[
(f f')(g g') + (f g)' = (f f')g + f(\int g')
\]

is satisfied. Then \((\mathcal{F}, \partial, \int)\) is an integro-differential algebra.

Examples of integro-differential algebras:

- \(\mathcal{F} = C^\infty(\mathbb{R}^n), \partial u = u_{x_i}, \int u = \int_0^{x_i} u(\xi) \, d\xi, \) partial for \(n > 1\)
- \(\mathcal{F} = C^\omega(D), \partial u = u', \int u = \int_0^z u(z) \, dz\)
- Holonomic functions \(\subset K[[x]]\)
- Matrix rings \((\mathcal{F}^{n \times n}, \partial, \int),\)
- Adjunctions \(K[x], K[x, e^x], K[x, \frac{1}{x}, \log x]\)
Alternative Characterizations

Let \((F, \partial)\) be a differential algebra and \(r\) a section of \(\partial\). Then the following statements are equivalent:

1. The structure \((F, r, \partial)\) is an integro-differential algebra.
2. We have \(E fg = E f E g\) for \(E := 1_F - r \circ \partial\).
3. We have \(J fJ g = fJ g, J J(f)g = J(f)g\) for \(J := r \circ \partial\).
4. One has \(I := \text{im} \ r \downarrow F\) while \(C := \ker \partial \leq F\).
5. Integration by parts \(r(f' r g) = f r g - r f g\), and opposite.
6. We have \((r f)(r g) = r(f r g) + r(g r f)\), and \(r\) is \(C\)-linear.

The structure \((F, r)\) in (6) is called Rota-Baxter-Algebra. We always have \(F = C \uplus I\) since \(1_F = E + J\). In \(F = C_\infty(R)\) have \(E(f) = f(0)\), so \(C = R, I = \{f | f(0) = 0\}\).
Proposition

Let \((\mathcal{F}, \partial)\) be a differential algebra and \(\int\) a section of \(\partial\). Then the following statements are equivalent:

1. The structure \((\mathcal{F}, \int, \partial)\) is an integro-differential algebra.

The structure \((\mathcal{F}, \int, \partial)\) in (6) is called Rota-Baxter-Algebra.

We always have \(\mathcal{F} = \mathbb{C} \bowtie \mathcal{I}\) since \(\mathcal{F} = \mathbb{E} + \mathcal{J}\). In \(\mathcal{F} = \mathbb{C}^\infty(\mathbb{R})\) have \(\mathcal{E}(f) = f(0)\), so \(\mathcal{C} = \mathbb{R}, \mathcal{I} = \{f| f(0) = 0\}\).

Ordinary \(\mathcal{F}\): Characters \(\varphi \in \mathcal{F}\) \(\leftrightarrow\) Integrals \(\int \varphi\) for \(\partial\).
Alternative Characterizations

**Proposition**

Let \((\mathcal{F}, \partial)\) be a differential algebra and \(\int\) a section of \(\partial\). Then the following statements are equivalent:

1. The structure \((\mathcal{F}, \int, \partial)\) is an integro-differential algebra.
2. We have \(E(fg) = E(f)E(g)\) for \(E := 1_{\mathcal{F}} - \int \circ \partial\).
Proposition

Let \((\mathcal{F}, \partial)\) be a differential algebra and \(\int\) a section of \(\partial\). Then the following statements are equivalent:

1. The structure \((\mathcal{F}, \int, \partial)\) is an integro-differential algebra.
2. We have \(E(fg) = E(f)E(g)\) for \(E := 1_{\mathcal{F}} - \int \circ \partial\).
3. We have \(J(fJ(g)) = fJ(g), J(J(f)g) = J(f)g\) for \(J := \int \circ \partial\).
Proposition

Let \((\mathcal{F}, \partial)\) be a differential algebra and \(\int\) a section of \(\partial\). Then the following statements are equivalent:

1. The structure \((\mathcal{F}, \int, \partial)\) is an integro-differential algebra.
2. We have \(E(fg) = E(f)E(g)\) for \(E := 1_{\mathcal{F}} - \int \circ \partial\).
3. We have \(J(fJ(g)) = fJ(g), J(J(f)g) = J(f)g\) for \(J := \int \circ \partial\).
4. One has \(\mathcal{I} := \text{im} \int \trianglelefteq \mathcal{F}\) while \(\mathcal{C} := \ker \partial \leq \mathcal{F}\).
Proposition

Let \((\mathcal{F}, \partial)\) be a differential algebra and \(\int\) a section of \(\partial\). Then the following statements are equivalent:

1. The structure \((\mathcal{F}, \int, \partial)\) is an integro-differential algebra.
2. We have \(E(fg) = E(f)E(g)\) for \(E := 1_{\mathcal{F}} - \int \circ \partial\).
3. We have \(J(fJ(g)) = fJ(g), J(J(f)g) = J(f)g\) for \(J := \int \circ \partial\).
4. One has \(\mathcal{I} := \text{im} \int \leq \mathcal{F}\) while \(\mathcal{C} := \ker \partial \leq \mathcal{F}\).
5. Integration by parts \(\int(f'\int g) = f\int g - \int fg\), and opposite.
Proposition

Let \((\mathcal{F}, \partial)\) be a differential algebra and \(\int\) a section of \(\partial\). Then the following statements are equivalent:

1. The structure \((\mathcal{F}, \int, \partial)\) is an integro-differential algebra.
2. We have \(E(fg) = E(f)E(g)\) for \(E := 1_{\mathcal{F}} - \int \circ \partial\).
3. We have \(J(f J(g)) = f J(g), J(J(f) g) = J(f) g\) for \(J := \int \circ \partial\).
4. One has \(\mathcal{I} := \text{im} \int \leq \mathcal{F}\) while \(\mathcal{C} := \ker \partial \leq \mathcal{F}\).
5. Integration by parts \(\int(f' \int g) = f \int g - \int fg\), and opposite.
6. We have \((\int f)(\int g) = \int(f \int g) + \int(g \int f)\), and \(\int\) is \(\mathcal{C}\)-linear.
Proposition

Let \((\mathcal{F}, \partial)\) be a differential algebra and \(\int\) a section of \(\partial\). Then the following statements are equivalent:

1. The structure \((\mathcal{F}, \int, \partial)\) is an integro-differential algebra.
2. We have \(E(fg) = E(f)E(g)\) for \(E := 1_{\mathcal{F}} - \int \circ \partial\).
3. We have \(J(fJ(g)) = fJ(g), J(J(f)g) = J(f)g\) for \(J := \int \circ \partial\).
4. One has \(\mathcal{I} := \text{im} \int \trianglelefteq \mathcal{F}\) while \(\mathcal{C} := \ker \partial \leq \mathcal{F}\).
5. Integration by parts \(\int (f' \int g) = f \int g - \int fg\), and opposite.
6. We have \((\int f)(\int g) = \int (f \int g) + \int (g \int f)\), and \(\int\) is \(\mathcal{C}\)-linear.

The structure \((\mathcal{F}, \int)\) in (6) is called Rota-Baxter-Algebra.
Proposition

Let \((\mathcal{F}, \partial)\) be a differential algebra and \(\int\) a section of \(\partial\). Then the following statements are equivalent:

1. The structure \((\mathcal{F}, \int, \partial)\) is an integro-differential algebra.
2. We have \(E(fg) = E(f)E(g)\) for \(E := 1_\mathcal{F} - \int \circ \partial\).
3. We have \(J(fJ(g)) = fJ(g), J(J(f)g) = J(f)g\) for \(J := \int \circ \partial\).
4. One has \(\mathcal{I} := \text{im} \int \subseteq \mathcal{F}\) while \(\mathcal{C} := \ker \partial \leq \mathcal{F}\).
5. Integration by parts \(\int (f'\int g) = f \int g - \int fg\), and opposite.
6. We have \((\int f)(\int g) = \int (f \int g) + \int (g \int f)\), and \(\int\) is \(\mathcal{C}\)-linear.

- The structure \((\mathcal{F}, \int)\) in (6) is called Rota-Baxter-Algebra.
- We always have \(\mathcal{F} = \mathcal{C} + \mathcal{I}\) since \(1_\mathcal{F} = E + J\).
Proposition

Let \((\mathcal{F}, \partial)\) be a differential algebra and \(\int\) a section of \(\partial\). Then the following statements are equivalent:

1. The structure \((\mathcal{F}, \int, \partial)\) is an integro-differential algebra.

2. We have \(E(fg) = E(f)E(g)\) for \(E := 1_{\mathcal{F}} - \int \circ \partial\).

3. We have \(J(fJ(g)) = fJ(g), J(J(f)g) = J(f)g\) for \(J := \int \circ \partial\).

4. One has \(\mathcal{I} := \text{im} \int \leq \mathcal{F}\) while \(\mathcal{C} := \ker \partial \leq \mathcal{F}\).

5. Integration by parts \(\int(f'\int g) = f\int g - \int fg\), and opposite.

6. We have \((\int f)(\int g) = \int(f\int g) + \int(g\int f)\), and \(\int\) is \(\mathcal{C}\)-linear.

- The structure \((\mathcal{F}, \int)\) in (6) is called Rota-Baxter-Algebra.
- We always have \(\mathcal{F} = \mathcal{C} + \mathcal{I}\) since \(1_{\mathcal{F}} = E + J\).
- In \(\mathcal{F} = \mathcal{C}^\infty(\mathbb{R})\) have \(E(f) = f(0)\), so \(\mathcal{C} = \mathbb{R}, \mathcal{I} = \{f \mid f(0) = 0\}\).
Proposition

Let \((\mathcal{F}, \partial)\) be a differential algebra and \(\int\) a section of \(\partial\). Then the following statements are equivalent:

1. The structure \((\mathcal{F}, \int, \partial)\) is an integro-differential algebra.
2. We have \(E(fg) = E(f)E(g)\) for \(E := 1_{\mathcal{F}} - \int \circ \partial\).
3. We have \(J(fJ(g)) = fJ(g), J(J(f)g) = J(f)g\) for \(J := \int \circ \partial\).
4. One has \(\mathcal{I} := \text{im} \int \subseteq \mathcal{F}\) while \(\mathcal{C} := \ker \partial \leq \mathcal{F}\).
5. Integration by parts \(\int (f' \int g) = f \int g - \int fg\), and opposite.
6. We have \((\int f)(\int g) = \int (f \int g) + \int (g \int f)\), and \(\int\) is \(\mathcal{C}\)-linear.

- The structure \((\mathcal{F}, \int)\) in (6) is called Rota-Baxter-Algebra.
- We always have \(\mathcal{F} = \mathcal{C} + \mathcal{I}\) since \(1_{\mathcal{F}} = E + J\).
- In \(\mathcal{F} = C^\infty(\mathbb{R})\) have \(E(f) = f(0)\), so \(\mathcal{C} = \mathbb{R}, \mathcal{I} = \{f \mid f(0) = 0\}\).
- Ordinary \(\mathcal{F}\): Characters \(\varphi \in \mathcal{F}^\bullet \leftrightarrow\) Integrals \(\int \varphi\) for \(\partial\).
Univariate Operator Ring

Definition and Theorem

Let \((F, \partial, r)\) be an ordinary integro-differential algebra. Then the ring of integro-differential operators \(F[\partial, r]\) is the \(K\)-algebra generated by \(\{\partial, r\} \cup F \cup F\cdot\) modulo the Gröbner basis below.

\[
\begin{align*}
\hat{f} \cdot f & \rightarrow \hat{f} \cdot f \\
\partial f & \rightarrow f \partial + f \partial \\
r f & \rightarrow f r - r f \\
\hat{\varphi} \cdot \varphi & \rightarrow \hat{\varphi} \cdot \varphi \\
\partial \varphi & \rightarrow 0 \\
r f & \rightarrow f - r f - f E \\
\varphi f & \rightarrow f \varphi \\
r \varphi & \rightarrow f r \varphi
\end{align*}
\]

Proposition

One has \(F[\partial, r] = F[\partial] \sqcup F[r] \sqcup (F\cdot)\), and the evaluation ideal \((F\cdot)\) is generated by \((F\cdot)\) as a left \(F\)-module.

Markus Rosenkranz

Differential Algebra for Boundary Problems
Definition and Theorem

Let \((\mathcal{F}, \partial, \int)\) be an ordinary integro-differential algebra. Then the ring of integro-differential operators \(\mathcal{F}[\partial, \int]\) is the \(K\)-algebra generated by \(\{\partial, \int\} \cup \mathcal{F} \cup \mathcal{F}^\bullet\) modulo the Gröbner basis below.

\[\begin{align*}
\tilde{f} & \rightarrow \tilde{f} \cdot f \\
\partial \tilde{f} & \rightarrow f \partial + f \partial \\
r \tilde{f} & \rightarrow \tilde{f} \cdot r - r \tilde{f} \\
\tilde{\phi} & \rightarrow \tilde{\phi} \cdot \partial \\
\partial \tilde{\phi} & \rightarrow 0 \\
r \tilde{\phi} & \rightarrow \tilde{\phi} \cdot r - r \tilde{\phi} \\
\tilde{f} \cdot \tilde{\phi} & \rightarrow \tilde{f} \cdot \tilde{\phi} \\
r \tilde{f} \tilde{\phi} & \rightarrow \tilde{f} r \tilde{\phi} \\
\tilde{f} \tilde{\phi} & \rightarrow \tilde{f} \tilde{\phi} \\
\end{align*}\]
**Definition and Theorem**

Let \((\mathcal{F}, \partial, \int)\) be an ordinary integro-differential algebra. Then the **ring of integro-differential operators** \(\mathcal{F}[\partial, \int]\) is the \(K\)-algebra generated by \(\{\partial, \int\} \cup \mathcal{F} \cup \mathcal{F}^\bullet\) modulo the Gröbner basis below.

\[
\begin{align*}
\tilde{f}f & \rightarrow \tilde{f} \cdot f \\
\tilde{\phi}\phi & \rightarrow \phi \\
\phi f & \rightarrow f^\phi \phi
\end{align*}
\]

\[
\begin{align*}
\partial f & \rightarrow f^\partial + f\partial \\
\partial \phi & \rightarrow 0 \\
\partial \int & \rightarrow 1
\end{align*}
\]

\[
\begin{align*}
\int f\int & \rightarrow f^\int \int - \int f^\int \\
\int f\partial & \rightarrow f - \int f^\partial - f^E E \\
\int f\phi & \rightarrow f^\int \phi
\end{align*}
\]
Let \((\mathcal{F}, \partial, \int)\) be an ordinary integro-differential algebra. Then the ring of integrro-differential operators \(\mathcal{F}[\partial, \int]\) is the \(K\)-algebra generated by \(\{\partial, \int\} \cup \mathcal{F} \cup \mathcal{F}^\bullet\) modulo the Gröbner basis below.

\[
\begin{align*}
\tilde{f} f & \rightarrow \tilde{f} \cdot f \\
\tilde{\varphi} \varphi & \rightarrow \varphi \\
\varphi f & \rightarrow f \varphi \\
\partial f & \rightarrow f^\partial + f \partial \\
\partial \varphi & \rightarrow 0 \\
\partial \int & \rightarrow 1 \\
\int f \int & \rightarrow f^\int \int - \int f^\int \\
\int f \partial & \rightarrow f - \int f^\partial - f^E E \\
\int f \varphi & \rightarrow f^\int \varphi
\end{align*}
\]

Proposition

One has \(\mathcal{F}[\partial, \int] = \mathcal{F}[\partial] \hat{+} \mathcal{F}[\int] \hat{+} (\mathcal{F}^\bullet)\), and the evaluation ideal \((\mathcal{F}^\bullet)\) is generated by \(|\mathcal{F}^\bullet|\) as a left \(\mathcal{F}\)-module.
Proposition

The normal forms of Stieltjes conditions \( |F \cdot \) are linear combinations of \( \phi \partial_i \) and global conditions \( \phi r_f \).

Example for \( F = C^\infty(\mathbb{R}) \) ist \( u \mapsto u(0) - u(1) + \int_0^1 \xi^2 u(\xi) d\xi \).

Stieltjes conditions appear in (some) applications. More importantly, they are inherently motivated (see below).

Classical two-point conditions as special case:

1. Only two evaluations \( L, R \).
2. No global parts.
3. Derivation order below that of differential equation.

Biintegro-differential algebras \( (F, \partial, A = r_L, B = -r_R) \):

Adjoint operators \( A \) and \( B \) relative to \( \langle f | g \rangle := (A + B)f \).

In \( F = C^\infty[a,b] \) we have \( A = r_x \) and \( B = r_1 x \).
Proposition

The normal forms of Stieltjes conditions \( \mathcal{F}^\bullet \) are linear combinations of local conditions \( \varphi \partial^i \) and global conditions \( \varphi \int f \).
Proposition

The normal forms of \textit{Stieltjes conditions} \(|\mathcal{F}^\bullet|\) are linear combinations of local conditions \(\varphi \partial^i\) and global conditions \(\varphi \int f\).

- Example for \(\mathcal{F} = C^\infty(\mathbb{R})\) ist \(u \mapsto u(0) - u(1) + \int_0^2 \xi^2 u(\xi) \, d\xi\).
Proposition

The normal forms of Stieltjes conditions $|\mathcal{F}^\bullet|$ are linear combinations of local conditions $\varphi \partial^i$ and global conditions $\varphi \int f$.

- Example for $\mathcal{F} = C^\infty(\mathbb{R})$ is $u \mapsto u(0) - u(1) + \int_0^2 \xi^2 u(\xi) \, d\xi$.
- Stieltjes conditions appear in (some) applications.
Proposition

The normal forms of **Stieltjes conditions** \(| \mathcal{F}^\bullet |\) are linear combinations of local conditions \(\varphi \partial^i\) and global conditions \(\varphi \int f\).

- Example for \(\mathcal{F} = C^\infty(\mathbb{R})\) ist \(u \mapsto u(0) - u(1) + \int_0^2 \xi^2 u(\xi) \, d\xi\).
- Stieltjes conditions appear in (some) applications.
- More importantly, they are inherently motivated (see below).
Stieltjes Conditions versus Two-Point Conditions

Proposition

The normal forms of Stieltjes conditions $|\mathcal{F}^\bullet|$ are linear combinations of local conditions $\varphi \partial^i$ and global conditions $\varphi \int f$.

- Example for $\mathcal{F} = C^\infty(\mathbb{R})$ is $u \mapsto u(0) - u(1) + \int_0^2 \xi^2 u(\xi) d\xi$.
- Stieltjes conditions appear in (some) applications.
- More importantly, they are inherently motivated (see below).

Classical two-point conditions as special case:
Proposition

The normal forms of \textbf{Stieltjes conditions} \( |F^\bullet| \) are linear combinations of local conditions \( \varphi \partial^i \) and global conditions \( \varphi \int f \).

- Example for \( F = C^\infty(\mathbb{R}) \) ist \( u \mapsto u(0) - u(1) + \int_0^2 \xi^2 u(\xi) \, d\xi \).
- Stieltjes conditions appear in (some) applications.
- More importantly, they are inherently motivated (see below).

Classical \textbf{two-point conditions} as special case:

1. Only two evaluations \( L, R \).
Proposition

The normal forms of Stieltjes conditions \(|\mathcal{F}^\bullet|\) are linear combinations of local conditions \(\varphi \partial_i\) and global conditions \(\varphi \int f\).

- Example for \(\mathcal{F} = C^\infty(\mathbb{R})\) is \(u \mapsto u(0) - u(1) + \int_0^2 \xi^2 u(\xi) \, d\xi\).
- Stieltjes conditions appear in (some) applications.
- More importantly, they are inherently motivated (see below).

Classical two-point conditions as special case:

1. Only two evaluations \(L, R\).
2. No global parts.
Proposition

The normal forms of Stieltjes conditions \( |\mathcal{F}^\bullet| \) are linear combinations of local conditions \( \varphi \partial^i \) and global conditions \( \varphi \int f \).

- Example for \( \mathcal{F} = C^\infty(\mathbb{R}) \) is \( u \mapsto u(0) - u(1) + \int_0^2 \xi^2 u(\xi) \, d\xi \).
- Stieltjes conditions appear in (some) applications.
- More importantly, they are inherently motivated (see below).

Classical two-point conditions as special case:

1. Only two evaluations \( L, R \).
2. No global parts.
3. Derivation order below that of differential equation.
Proposition

The normal forms of Stieltjes conditions $|F^\bullet|$ are linear combinations of local conditions $\varphi \partial^i$ and global conditions $\varphi \int f$.

- Example for $F = C^\infty(\mathbb{R})$ is $u \mapsto u(0) - u(1) + \int_0^2 \xi^2 u(\xi) \, d\xi$.
- Stieltjes conditions appear in (some) applications.
- More importantly, they are inherently motivated (see below).

Classical two-point conditions as special case:

1. Only two evaluations $L, R$.
2. No global parts.
3. Derivation order below that of differential equation.

Biintegro-differential algebras $(F, \partial, A = \int_L, B = -\int_R)$:
Stieltjes Conditions versus Two-Point Conditions

Proposition

The normal forms of Stieltjes conditions $\mathcal{F}$ are linear combinations of local conditions $\varphi \partial^i$ and global conditions $\varphi \int f$.

- Example for $\mathcal{F} = C^\infty(\mathbb{R})$ ist $u \mapsto u(0) - u(1) + \int_0^1 \xi^2 u(\xi) \, d\xi$.
- Stieltjes conditions appear in (some) applications.
- More importantly, they are inherently motivated (see below).

Classical two-point conditions as special case:

1. Only two evaluations $L, R$.
2. No global parts.
3. Derivation order below that of differential equation.

Biintegro-differential algebras $(\mathcal{F}, \partial, A = \int_L, B = -\int_R)$:

- Adjoint operators $A$ and $B$ relative to $\langle f | g \rangle := (A + B)f$. 
Proposition

The normal forms of \textbf{Stieltjes conditions }$|\mathcal{F}^\bullet|$ are linear combinations of local conditions $\varphi \partial^i$ and global conditions $\varphi \int f$.

- Example for $\mathcal{F} = C^\infty(\mathbb{R})$ is $u \mapsto u(0) - u(1) + \int_0^2 \xi^2 u(\xi) \, d\xi$.
- Stieltjes conditions appear in (some) applications.
- More importantly, they are inherently motivated (see below).

**Classical two-point conditions** as special case:

1. Only two evaluations $L, R$.
2. No global parts.
3. Derivation order below that of differential equation.

**Biintegro-differential algebras** $(\mathcal{F}, \partial, A = \int_L, B = -\int_R)$:

- Adjoint operators $A$ and $B$ relative to $\langle f | g \rangle := (A + B)f$.
- In $\mathcal{F} = C^\infty[a, b]$ we have $A = \int_0^x$ and $B = \int_1^x$. 
Concrete Boundary Problems for LODEs

Definition

A (concrete) boundary problem is a pair \((T, B)\) with \(T \in \mathbb{F}\left[\partial\right]\) a monic differential operator and \(B \leq \mathbb{F}^\ast\) a boundary space of Stieltjes conditions.

Concrete boundary problems form a submonoid of \(\mathbb{B}_{n,\text{Prob}}(\mathbb{F})\).

Regularity implies \(\text{ord} T = \dim B\), matrix test applicable.

Concrete regular problems submonoid of \(\mathbb{B}_{n,\text{Prob}}^\ast(\mathbb{F})\).

Distinguish regular from well-posed.

Theorem

Relative to a given fundamental system \(u_1, \ldots, u_n\) of \(T\), we can compute the Green's operator of \((T, B)\) as an element of \(\mathbb{F}[\partial, r]\).

Two-point problems: Normal form of 

\[ G \sim = \text{Green's function} \]

\[ Gf = \int_a^b g(x, \xi) f(\xi) d\xi \]
A (concrete) **boundary problem** is a pair \((T, B)\) with \(T \in \mathcal{F}[\partial]\) a monic differential operator and \(B \leq \mathcal{F}^*\) a boundary space of Stieltjes conditions.
Concrete Boundary Problems for LODEs

**Definition**

A (concrete) **boundary problem** is a pair \((T, B)\) with \(T \in \mathcal{F}[\partial]\) a monic differential operator and \(B \leq \mathcal{F}^*\) a boundary space of Stieltjes conditions.

- Concrete boundary problems form a submonoid of \(\text{BnProb}(\mathcal{F})\).
Concrete Boundary Problems for LODEs

**Definition**

A (concrete) **boundary problem** is a pair $(T, B)$ with $T \in \mathcal{F}[\partial]$ a monic differential operator and $B \leq \mathcal{F}^*$ a boundary space of Stieltjes conditions.

- Concrete boundary problems form a submonoid of $\text{BnProb}(\mathcal{F})$.
- Regularity implies $\text{ord } T = \text{dim } B$, matrix test applicable.
A (concrete) **boundary problem** is a pair \((T, B)\) with \(T \in \mathcal{F}[\partial]\) a monic differential operator and \(B \leq \mathcal{F}^*\) a boundary space of Stieltjes conditions.

- Concrete boundary problems form a submonoid of \(\text{BnProb} (\mathcal{F})\).
- Regularity implies \(\text{ord } T = \dim B\), matrix test applicable.
- Concrete regular problems submonoid of \(\text{BnProb}^* (\mathcal{F})\).
Concrete Boundary Problems for LODEs

**Definition**

A (concrete) **boundary problem** is a pair \((T, B)\) with \(T \in \mathcal{F}[\partial]\) a monic differential operator and \(B \leq \mathcal{F}^*\) a boundary space of Stieltjes conditions.

- Concrete boundary problems form a submonoid of \(\text{BnProb}(\mathcal{F})\).
- Regularity implies \(\text{ord } T = \dim B\), matrix test applicable.
- Concrete regular problems submonoid of \(\text{BnProb}^*(\mathcal{F})\).
- Distinguish regular from well-posed.
Concrete Boundary Problems for LODEs

**Definition**

A (concrete) **boundary problem** is a pair \((T, B)\) with \(T \in \mathcal{F}[\partial]\) a monic differential operator and \(B \leq \mathcal{F}^*\) a boundary space of Stieltjes conditions.

- Concrete boundary problems form a submonoid of \(\text{BnProb}(\mathcal{F})\).
- Regularity implies \(\text{ord } T = \dim B\), matrix test applicable.
- Concrete regular problems submonoid of \(\text{BnProb}^*(\mathcal{F})\).
- Distinguish regular from well-posed.

**Theorem**

Relative to a given fundamental system \(u_1, \ldots, u_n\) of \(T\), we can compute the **Green's operator** of \((T, B)\) as an element of \(\mathcal{F}[\partial, \int]\).
Concrete Boundary Problems for LODEs

### Definition

A (concrete) **boundary problem** is a pair \((T, B)\) with \(T \in \mathcal{F}[\partial]\) a monic differential operator and \(B \leq \mathcal{F}^*\) a boundary space of Stieltjes conditions.

- Concrete boundary problems form a submonoid of \(\text{BnProb}(\mathcal{F})\).
- Regularity implies \(\text{ord } T = \dim B\), matrix test applicable.
- Concrete regular problems submonoid of \(\text{BnProb}^*(\mathcal{F})\).
- Distinguish regular from well-posed.

### Theorem

Relative to a given fundamental system \(u_1, \ldots, u_n\) of \(T\), we can compute the **Green’s operator** of \((T, B)\) as an element of \(\mathcal{F}[\partial, \int]\).

**Two-point problems**: Normal form of \(G \cong \text{Green’s function } g(x, \xi)\)

\[
Gf = \int_a^b g(x, \xi) f(\xi) \, d\xi
\]
LODE Example Revisited

Recall previous two-point problem (taking $a = 0$, $b = 1$ for simplicity):

\[
\begin{align*}
  u'' &= f \\
  u(0) + u(1) &= \rho, \quad u'(1) - u(1) = \sigma
\end{align*}
\]
LODE Example Revisited

Recall previous two-point problem (taking $a = 0$, $b = 1$ for simplicity):

\[ u'' = f \]
\[ u(0) + u(1) = \rho, \quad u'(1) - u(1) = \sigma \]

Underlying boundary problem \((D^2, [L + R, RD - R])\)
Recall previous two-point problem (taking $a = 0, b = 1$ for simplicity):

\[
\begin{align*}
  u'' &= f \\
  u(0) + u(1) &= \rho, \quad u'(1) - u(1) = \sigma
\end{align*}
\]

Underlying boundary problem \((D^2, [L + R, RD - R])\)
Kernel projector \(P = (R - RD) + X(L - R + 2RD)\)
Recall previous two-point problem (taking $a = 0, b = 1$ for simplicity):

\[
\begin{align*}
\frac{d^2 u}{dx^2} &= f \\
u(0) + u(1) &= \rho, \quad u'(1) - u(1) = \sigma
\end{align*}
\]

Underlying boundary problem $(D^2, [L + R, RD - R])$

Kernel projector $P = (R - RD) + X(L - R + 2RD)$

Green’s operator $G = (1 - P)A^2 = BX - XB - XAX - XBX$
Recall previous two-point problem (taking $a = 0, b = 1$ for simplicity):

\[
\begin{align*}
    u'' &= f \\
    u(0) + u(1) &= \rho, \\
    u'(1) - u(1) &= \sigma
\end{align*}
\]

Underlying boundary problem ($D^2, [L + R, RD - R]$)

Kernel projector $P = (R - RD) + X(L - R + 2RD)$

Green’s operator $G = (1 - P)A^2 = BX - XB - XAX - XBX$

Green’s function $g(x, \xi) = \begin{cases} 
    -x\xi & \text{if } \xi \leq x \\
    \xi - x - x\xi & \text{if } \xi \geq x
\end{cases}$
Recall previous two-point problem (taking $a = 0, b = 1$ for simplicity):

\[
\begin{align*}
&u'' = f \\
&u(0) + u(1) = \rho, u'(1) - u(1) = \sigma
\end{align*}
\]

Underlying boundary problem $(D^2, [L + R, RD - R])$

Kernel projector $P = (R - RD) + X(L - R + 2RD)$

Green’s operator $G = (1 - P)A^2 = BX - XB - XAX - XBX$

Green’s function $g(x, \xi) = \begin{cases} 
-x\xi & \text{if } \xi \leq x \\
\xi - x - x\xi & \text{if } \xi \geq x
\end{cases}$

For completeness:

- Semi-homogeneous Green’s operator $H(\rho, \sigma) = (\rho + 2\sigma)x - \sigma$
LODE Example Revisited

Recall previous two-point problem (taking $a = 0, b = 1$ for simplicity):

\[
\begin{align*}
  u'' &= f \\
  u(0) + u(1) &= \rho, \ u'(1) - u(1) &= \sigma
\end{align*}
\]

Underlying boundary problem $\left( D^2, [L + R, RD - R] \right)$

Kernel projector $P = (R - RD) + X(L - R + 2RD)$

**Green’s operator**

$G = (1 - P)A^2 = BX - XB - XAX - XBX$

**Green’s function**

$g(x, \xi) = \begin{cases} 
  -x\xi & \text{if } \xi \leq x \\
  \xi - x - x\xi & \text{if } \xi \geq x 
\end{cases}$

For completeness:

- Semi-homogeneous Green’s operator $H(\rho, \sigma) = (\rho + 2\sigma)x - \sigma$

- For LODEs, determining $H$ is trivial (assuming fundamental system).
Third-Order Example

\((T, \mathcal{B}) = (D^3 - e^x D^2 - 2D^2 - D + e^x + 2, [L, R, RD])\)
(T, B) = (D^3 − e^x D^2 − 2D^2 − D + e^x + 2, [L, R, RD])

Classical Notation:

\[
\begin{align*}
    u''' - (e^x + 2) u'' - u' + (e^x + 2) u(x) &= f \\
u(0) = u(1) = u'(1) &= 0
\end{align*}
\]
Third-Order Example

\[(T, B) = (D^3 - e^x D^2 - 2D^2 - D + e^x + 2, [L, R, RD])\]

Classical Notation:

\[
\begin{align*}
    u''' - (e^x + 2) u'' - u' + (e^x + 2) u(x) &= f \\
    u(0) = u(1) = u'(1) &= 0
\end{align*}
\]

Green’s Operator:

\[
G = (e^{e^x - x} - e^{e^x}) B \left( e^{-e^x} + 2e^{-e} e(x) \right) + \sinh(x) B \left( 1 + 2e(x) \right) \\
+ \left( 2e^{e^x - e} (e^{-x} - 1) - (e - 1)^2 e^{-x} + 2 \sinh(x) \right) A e(x)
\]
Third-Order Example

\[(T, B) = (D^3 - e^x D^2 - 2D^2 - D + e^x + 2, [L, R, RD])\]

Classical Notation:

\[
\begin{align*}
&u''' - (e^x + 2)u'' - u' + (e^x + 2)u(x) = f \\
&u(0) = u(1) = u'(1) = 0
\end{align*}
\]

Green’s Operator:

\[
G = (e^{e^x - x} - e^{e^x}) B (e^{-e^x} + 2e^{-e} e(x)) + \sinh(x) B (1 + 2e(x)) \\
+ (2e^{e^x - e}(e^{-x} - 1) - (e - 1)^2 e^{-x} + 2 \sinh(x)) A e(x)
\]

Green’s Function: \( g(x, \xi) = \)

\[
\begin{cases} 
(2e^{e^x - e}(e^{-x} - 1) - (e - 1)^2 e^{-x} + 2 \sinh(x)) e^{2\xi} e(\xi) \\
(e^{e^x - x} - e^{e^x}) (e^{-e^x} + 2e^{-e} e(\xi)) + \sinh(x) e^{2\xi} (1 + 2e(\xi))
\end{cases}
\]

\[
e(t) := -\frac{1}{2} \left( \frac{e^t - 1}{e - 1} \right)^2
\]
Factorization can always be lifted.

Simplest Example: \[(D^2, [L,R]) = (D, [F]) \cdot (D^2, [L])\] with \(F := r_0 b \cdot \) or \(\frac{d^2u}{dx^2} = f \quad u(a) = u(b) = 0 = u'(a) = u'(b) = 0\)

Fourth-Order Example (Kamke 4.2): \[(D^4 + 4, [L,R,LD,RD]) = (D^2 - 2\i, [F_e(\i - 1)x, F_e(1 - \i)x]) \cdot (D^2 + 2\i, [L,R])\] or \(\frac{d^4u}{dx^4} + 4u = f \quad u(a) = u(b) = u'(a) = u'(b) = 0 = u'' - 2\i u = f \quad r_0 b e^((\i - 1)\xi) u(\xi) d\xi = r_0 b e^{(1 - \i)\xi} u(\xi) d\xi = 0 \cdot u'' + 2\i u = f \quad u(a) = u(b) = 0\)
Factorization can always be lifted.

Simplest Example:

$$(D^2, [L, R]) = (D, [F]) \cdot (D, [L]) \text{ with } F := \int_{a}^{b} \cdot$$
Factorization can always be lifted.

Simplest Example:

\[(D^2, [L, R]) = (D, [F]) \cdot (D, [L]) \text{ with } F := \int_a^b\]

or

\[
\begin{align*}
  u'' &= f \\
  u(a) &= u(b) = 0
\end{align*}
\]

\[
\begin{align*}
  u' &= f \\
  \int_a^b u(\xi) \, d\xi &= 0
\end{align*}
\]

\[
\begin{align*}
  u'(a) &= 0
\end{align*}
\]
Factorization of Ordinary Boundary Problems

Factorization can always be lifted.

Simplest Example:

\[(D^2, [L, R]) = (D, [F]) \cdot (D, [L]) \text{ with } F := \int_a^b\]

or

\[
\begin{align*}
 u'' &= f \\
 u(a) = u(b) &= 0
\end{align*}
\]

= \[
\begin{align*}
 u' &= f \\
 \int_a^b u(\xi) \, d\xi &= 0
\end{align*}
\]

\[
\begin{align*}
 u' &= f \\
 u(a) &= 0
\end{align*}
\]

Fourth-Order Example (Kamke 4.2):

\[(D^4 + 4, [L, R, LD, RD]) = (D^2 - 2i, [Fe^{(i-1)x}, Fe^{(1-i)x}]) \cdot (D^2 + 2i, [L, R])\]
Factorization can always be lifted.

Simplest Example:

\[(D^2, [L, R]) = (D, [F]) \cdot (D, [L]) \text{ with } F := \int_a^b\]

or

\[
\begin{align*}
    u'' &= f \\
    u(a) &= u(b) = 0
\end{align*}
\]

\[
\begin{align*}
    u' &= f \\
    \int_a^b u(\xi) \, d\xi &= 0
\end{align*}
\]

\[
\begin{align*}
    u' &= f \\
    u(a) &= 0
\end{align*}
\]

Fourth-Order Example (Kamke 4.2):

\[(D^4 + 4, [L, R, LD, RD]) = (D^2 - 2i, [Fe^{(i-1)x}, Fe^{(1-i)x}]) \cdot (D^2 + 2i, [L, R])\]

or

\[
\begin{align*}
    u''' + 4u &= f \\
    u(a) &= u(b) = u'(a) = u'(b) = 0
\end{align*}
\]

or

\[
\begin{align*}
    u'' - 2i u &= f \\
    \int_a^b e^{(i-1)\xi} u(\xi) \, d\xi &= \int_a^b e^{(1-i)\xi} u(\xi) \, d\xi = 0
\end{align*}
\]

\[
\begin{align*}
    u'' + 2i u &= f \\
    u(a) &= u(b) = 0
\end{align*}
\]
1. Abstract Boundary Problems
2. Ordinary Integro-Differential Operators
3. Partial Integro-Differential Operators
4. Conclusion
Basic Example: Smooth Functions

For simplicity first omit $\partial_x, \partial_y, \ldots$; only consider $r_x, r_y, \ldots$.

Admit all smooth functions $f(x,y,\ldots) \in F$ to be operated on.

Take multipliers $g(x,y,\ldots)$ from a suitably nice subalgebra $G \subseteq F$.

Allow all substitutions $f(x,y,\ldots) \mapsto f(ax + by, cx + dy, \ldots)$ for $a,b,c,d \in \mathbb{R}$.

For convenience view $F$ as filtered algebra $F = \bigcup_{n=0}^{\infty} F_n$ with $C^\infty(\mathbb{R}^n)$ and natural injections $C^\infty(\mathbb{R}^n) \hookrightarrow C^\infty(\mathbb{R}^{n+1})$.

Similarly use filtered monoid $M(\mathbb{R}) = \bigcup_{n=1}^{\infty} M_n(\mathbb{R})$ where $M_n(\mathbb{R})$ are near-identity matrices with injections $M \hookrightarrow (I_{n+0})$. 
Basic Example: Smooth Functions

For simplicity first omit $\partial_x, \partial_y, \ldots$; only consider $\int^x, \int^y, \ldots$. 
Basic Example: Smooth Functions

- For simplicity first omit $\partial_x, \partial_y, \ldots$; only consider $\int^x, \int^y, \ldots$.
- Admit all smooth functions $f(x, y, \ldots) \in \mathcal{F}$ to be operated on.
Basic Example: Smooth Functions

- For simplicity first omit $\partial_x, \partial_y, \ldots$; only consider $\int^x, \int^y, \ldots$.
- Admit all smooth functions $f(x, y, \ldots) \in \mathcal{F}$ to be operated on.
- Take multipliers $g(x, y, \ldots)$ from a suitably nice subalgebra $\mathcal{G} \subseteq \mathcal{F}$. 

- Allow all substitutions $f(x, y, \ldots) \mapsto f(ax + by, cx + dy, \ldots)$ for $a, b, c, d \in \mathbb{R}$.
- For convenience view $\mathcal{F}$ as filtered algebra $\mathcal{F} = \bigcup_{n=0}^{\infty} \mathcal{F}_n = \bigcup_{n=0}^{\infty} C^\infty(\mathbb{R}^n)$ with $C^\infty(\mathbb{R}^0) = \mathbb{R}$ and natural injections $C^\infty(\mathbb{R}^n) \hookrightarrow C^\infty(\mathbb{R}^{n+1})$.

Similarly use filtered monoid $\mathcal{M}(\mathbb{R}) = \bigcup_{n=1}^{\infty} \mathcal{M}_n(\mathbb{R})$ where $\mathcal{M}_n(\mathbb{R})$ are near-identity matrices with injections $\mathcal{M} \hookrightarrow \left( I_{\mathbb{R}^n} \right)$. 

Markus Rosenkranz

Differential Algebra for Boundary Problems
Basic Example: Smooth Functions

- For simplicity first omit $\partial_x, \partial_y, \ldots$; only consider $\int^x, \int^y, \ldots$.
- Admit all smooth functions $f(x, y, \ldots) \in \mathcal{F}$ to be operated on.
- Take multipliers $g(x, y, \ldots)$ from a suitably nice subalgebra $\mathcal{G} \subseteq \mathcal{F}$.
- Allow all substitutions $f(x, y, \ldots) \mapsto f(ax + by, cx + dy, \ldots)$ for $a, b, c, d \in \mathbb{R}$.
Basic Example: Smooth Functions

- For simplicity first omit $\partial_x, \partial_y, \ldots$; only consider $\int^x, \int^y, \ldots$.
- Admit all smooth functions $f(x, y, \ldots) \in \mathcal{F}$ to be operated on.
- Take multipliers $g(x, y, \ldots)$ from a suitably nice subalgebra $\mathcal{G} \subseteq \mathcal{F}$.
- Allow all substitutions $f(x, y, \ldots) \mapsto f(ax + by, cx + dy, \ldots)$ for $a, b, c, d \in \mathbb{R}$.

For convenience view $\mathcal{F}$ as filtered algebra

$$\mathcal{F} = \bigcup_{n=0}^{\infty} \mathcal{F}_n := \bigcup_{n=0}^{\infty} C^\infty(\mathbb{R}^n)$$

with $C^\infty(\mathbb{R}^0) := \mathbb{R}$ and natural injections $C^\infty(\mathbb{R}^n) \hookrightarrow C^\infty(\mathbb{R}^{n+1})$. 
Basic Example: Smooth Functions

- For simplicity first omit $\partial_x, \partial_y, \ldots$; only consider $\int^x, \int^y, \ldots$.
- Admit all smooth functions $f(x, y, \ldots) \in \mathcal{F}$ to be operated on.
- Take multipliers $g(x, y, \ldots)$ from a suitably nice subalgebra $\mathcal{G} \subseteq \mathcal{F}$.
- Allow all substitutions $f(x, y, \ldots) \mapsto f(ax + by, cx + dy, \ldots)$ for $a, b, c, d \in \mathbb{R}$.

For convenience view $\mathcal{F}$ as filtered algebra

$$
\mathcal{F} = \bigcup_{n=0}^{\infty} \mathcal{F}_n := \bigcup_{n=0}^{\infty} C^\infty(\mathbb{R}^n)
$$

with $C^\infty(\mathbb{R}^0) := \mathbb{R}$ and natural injections $C^\infty(\mathbb{R}^n) \hookrightarrow C^\infty(\mathbb{R}^{n+1})$.

Similarly use filtered monoid

$$
\mathcal{M}(\mathbb{R}) = \bigcup_{n=1}^{\infty} \mathcal{M}_n(\mathbb{R})
$$

where $\mathcal{M}_n(\mathbb{R})$ are near-identity matrices with injections $M \hookrightarrow \begin{pmatrix} I_n & 0 \\ 0 & 1 \end{pmatrix}$. 
Action of Integrals and Substitutions

Write $r_x$:

$F \to F$ for Rota-Baxter operator $f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots) \mapsto \int_{x_i^0} f(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots) \, d\xi$.

Given $M \in M(R)$ write $M^* f := g$ with $g(x_1, x_2, \ldots) := f(\sum_i M^1_i x_i, \sum_i M^2_i x_i, \ldots)$, for contravariant monoid action $M(R) \times F \to F$ via algebra morphisms.

Hence note $(MN)^* = N^* M^*$.

But what about $r_x^* M^*$?
Write $\int^{x_i} : \mathcal{F} \to \mathcal{F}$ for Rota-Baxter operator

$$f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots) \mapsto \int_0^{x_i} f(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots) \, d\xi.$$
Write $\int^{x_i} : \mathcal{F} \to \mathcal{F}$ for Rota-Baxter operator

$$f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots) \mapsto \int_0^{x_i} f(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots) \, d\xi.$$ 

Given $M \in \mathcal{M}(\mathbb{R})$ write $M^* f =: g$ with

$$g(x_1, x_2, \ldots) := f\left( \sum_i M_{1i} x_i, \sum_i M_{2i} x_i, \ldots \right),$$ 

for contravariant monoid action $\mathcal{M}(\mathbb{R}) \times \mathcal{F} \to \mathcal{F}$ via algebra morphisms.
Write $\int^{x_i} : \mathcal{F} \to \mathcal{F}$ for Rota-Baxter operator

$$f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots) \mapsto \int_0^{x_i} f(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots) \, d\xi.$$ 

Given $M \in \mathcal{M}(\mathbb{R})$ write $M^* f =: g$ with

$$g(x_1, x_2, \ldots) := f\left(\sum_i M_{1i} x_i, \sum_i M_{2i} x_i, \ldots\right),$$

for contravariant monoid action $\mathcal{M}(\mathbb{R}) \times \mathcal{F} \to \mathcal{F}$ via algebra morphisms. Hence note $(MN)^* = N^* M^*$. 
Write $\int^{x_i} : \mathcal{F} \to \mathcal{F}$ for Rota-Baxter operator

$$f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots) \mapsto \int_0^{x_i} f(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots) \, d\xi.$$ 

Given $M \in \mathcal{M}(\mathbb{R})$ write $M^* f =: g$ with

$$g(x_1, x_2, \ldots) := f\left(\sum_i M_{1i} x_i, \sum_i M_{2i} x_i, \ldots\right),$$

for contravariant monoid action $\mathcal{M}(\mathbb{R}) \times \mathcal{F} \to \mathcal{F}$ via algebra morphisms. Hence note $(MN)^* = N^* M^*$. But what about $\int^{x_i} M^*$?
Notation for Special Matrices

Evaluation at $x_i$ :

$$E_i = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \leftarrow i \end{pmatrix}$$
Notation for Special Matrices

Evaluation at $x_i$:

$$E_i = \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & \cdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix}$$

Transvection for $v \in K^{n-1}$:

$$T_i(v) = \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ v_1 & \cdots & v_{i-1} & 1 \\ \vdots & \ddots & \ddots & \vdots \\ v_i & \cdots & 1 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ v_{i+1} & \cdots & v_n & 1 \\ \end{pmatrix}$$
Notation for Special Matrices

Evaluation at $x_i$:

$$E_i = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \leftarrow i \end{pmatrix}$$

Transvection for $v \in K^{n-1}$:

$$T_i(v) = \begin{pmatrix} 1 \\ \vdots \\ v_1 \end{pmatrix} \begin{pmatrix} \vdots & \cdots & \vdots & \cdots & \vdots \\ v_{i-1} & 1 & v_{i+1} & \cdots & v_n \end{pmatrix} 1$$

Eliminant for $w \in K^{n-i}$:

$$L_i(w) = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ w_{i+1} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ w_n \end{pmatrix} 1$$
Substitutive Algebras

Definition

An ascending $K$-algebra $(F_n)$ is called substitutive if it has a straight contravariant monoid action of $M(K)$ such that $M^* F_n \subseteq F_n$ for all $M \in M_n(K)$ and $E^* n (F_n) \subseteq F_{n-1}$. We write $F = \lim \rightarrow F_n$.

In detail $^*: M(K) \rightarrow \text{Hom}_{\text{Alg}}(F)$ with $I^* = 1_{F_n}$, $(MN)^* = N^* M^*$.

Straightness means $M^* f = M^* \llcorner f$ for all $M \in M(K)$ and $f \in F_n$.

Define dependence hierarchy:

$F_\alpha = \{ f \in F | \pi^* f \in F_k \}$ for $\alpha = (\alpha_1, \ldots, \alpha_k) \subset \mathbb{N}$.

$F_\alpha = \bigcup_{n=1}^{\infty} F(\alpha_1, \ldots, \alpha_n)$ for arbitrary $\alpha \subset \mathbb{N}$ by monotonicity.

$\rightarrow$ Complete complemented lattice: $(F_\alpha)$ with $F_\alpha \uplus F_\beta = F_\alpha \cup \beta$, $F_\alpha \cap F_\beta = F_\alpha \cap \beta$, $F_\emptyset = K$, $F_N = F$, $F'_\alpha = F_N \setminus \alpha$. 

Markus Rosenkranz

Differential Algebra for Boundary Problems
An ascending \( K \)-algebra \((\mathcal{F}_n)\) is called **substitutive** if it has a straight contravariant monoid action of \( \mathcal{M}(K) \) such that \( M^*(\mathcal{F}_n) \subseteq \mathcal{F}_n \) for all \( M \in \mathcal{M}_n(K) \) and \( E_n^*(\mathcal{F}_n) \subseteq \mathcal{F}_{n-1} \). We write \( \mathcal{F} = \lim_{\longrightarrow} \mathcal{F}_n \).
Substitutive Algebras

Definition

An ascending $K$-algebra $(\mathcal{F}_n)$ is called **substitutive** if it has a straight contravariant monoid action of $\mathcal{M}(K)$ such that $M^*(\mathcal{F}_n) \subseteq \mathcal{F}_n$ for all $M \in \mathcal{M}_n(K)$ and $E_n^*(\mathcal{F}_n) \subseteq \mathcal{F}_{n-1}$. We write $\mathcal{F} = \lim \mathcal{F}_n$.

In detail $*: \mathcal{M}(K) \rightarrow \text{Hom}_{\text{Alg}}(\mathcal{F})$ with $I^* = 1_\mathcal{F}$, $(MN)^* = N^*M^*$. 
**Substitutive Algebras**

**Definition**

An ascending $K$-algebra $(\mathcal{F}_n)$ is called **substitutive** if it has a straight contravariant monoid action of $\mathcal{M}(K)$ such that $M^*(\mathcal{F}_n) \subseteq \mathcal{F}_n$ for all $M \in \mathcal{M}_n(K)$ and $E_n^*(\mathcal{F}_n) \subseteq \mathcal{F}_{n-1}$. We write $\mathcal{F} = \varprojlim \mathcal{F}_n$.

In detail $\ast : \mathcal{M}(K) \rightarrow \operatorname{Hom}_{\text{Alg}}(\mathcal{F})$ with $I^* = 1_\mathcal{F}$, $(MN)^* = N^* M^*$. **Straightness** means $M^* f = M^*_{\downarrow} f$ for all $M \in \mathcal{M}(K)$ and $f \in \mathcal{F}_n$. 

---

*Markus Rosenkranz*

*Differential Algebra for Boundary Problems*
An ascending $K$-algebra $(\mathcal{F}_n)$ is called **substitutive** if it has a straight contravariant monoid action of $\mathcal{M}(K)$ such that $M^*(\mathcal{F}_n) \subseteq \mathcal{F}_n$ for all $M \in \mathcal{M}_n(K)$ and $E^*_n(\mathcal{F}_n) \subseteq \mathcal{F}_{n-1}$. We write $\mathcal{F} = \varinjlim \mathcal{F}_n$.

In detail $*: \mathcal{M}(K) \to \text{Hom}_{\text{Alg}}(\mathcal{F})$ with $I^* = 1_{\mathcal{F}}$, $(MN)^* = N^*M^*$.

**Straightness** means $M^*f = M^*_{\downarrow n}f$ for all $M \in \mathcal{M}(K)$ and $f \in \mathcal{F}_n$.

Define **dependence hierarchy**:

$$\mathcal{F}_\alpha = \{ f \in \mathcal{F} \mid \pi^*f \in \mathcal{F}_k \} \text{ for } \alpha = (\alpha_1, \ldots, \alpha_k) \subset \mathbb{N}$$
Substitutive Algebras

**Definition**

An ascending $K$-algebra $(\mathcal{F}_n)$ is called **substitutive** if it has a straight contravariant monoid action of $\mathcal{M}(K)$ such that $M^*(\mathcal{F}_n) \subseteq \mathcal{F}_n$ for all $M \in \mathcal{M}_n(K)$ and $E_n^*(\mathcal{F}_n) \subseteq \mathcal{F}_{n-1}$. We write $\mathcal{F} = \varinjlim \mathcal{F}_n$.

In detail $\ast : \mathcal{M}(K) \to \text{Hom}_{\text{Alg}}(\mathcal{F})$ with $I^* = 1_\mathcal{F}$, $(MN)^* = N^*M^*$.

**Straightness** means $M^*f = M^*_n f$ for all $M \in \mathcal{M}(K)$ and $f \in \mathcal{F}_n$.

Define **dependence hierarchy**:

\[ \mathcal{F}_\alpha = \{ f \in \mathcal{F} \mid \pi^* f \in \mathcal{F}_k \} \text{ for } \alpha = (\alpha_1, \ldots, \alpha_k) \subseteq \mathbb{N} \]

\[ \mathcal{F}_\alpha = \bigcup_{n=1}^\infty \mathcal{F}(\alpha_1, \ldots, \alpha_n) \text{ for arbitrary } \alpha \subseteq \mathbb{N} \text{ by monotonicity} \]
Substitutive Algebras

Definition

An ascending $K$-algebra $(\mathcal{F}_n)$ is called **substitutive** if it has a straight contravariant monoid action of $\mathcal{M}(K)$ such that $M^*(\mathcal{F}_n) \subseteq \mathcal{F}_n$ for all $M \in \mathcal{M}_n(K)$ and $E^*_n(\mathcal{F}_n) \subseteq \mathcal{F}_{n-1}$. We write $\mathcal{F} = \lim \downarrow \mathcal{F}_n$.

In detail $*: \mathcal{M}(K) \to \text{Hom}_{\text{Alg}}(\mathcal{F})$ with $I^* = 1_{\mathcal{F}}$, $(MN)^* = N^*M^*$.

**Straightness** means $M^*f = M^*_n f$ for all $M \in \mathcal{M}(K)$ and $f \in \mathcal{F}_n$.

Define **dependence hierarchy**:

$\mathcal{F}_\alpha = \{ f \in \mathcal{F} \mid \pi^* f \in \mathcal{F}_k \}$ for $\alpha = (\alpha_1, \ldots, \alpha_k) \subseteq \mathbb{N}$

$\mathcal{F}_\alpha = \bigcup_{n=1}^{\infty} \mathcal{F}_{(\alpha_1, \ldots, \alpha_n)}$ for arbitrary $\alpha \subseteq \mathbb{N}$ by monotonicity

$\rightarrow$ **Complete complemented lattice**:

$(\mathcal{F}_\alpha)$ with $\mathcal{F}_\alpha \sqcup \mathcal{F}_\beta = \mathcal{F}_{\alpha \cup \beta}$, $\mathcal{F}_\alpha \sqcap \mathcal{F}_\beta = \mathcal{F}_{\alpha \cap \beta}$

$\mathcal{F}_\emptyset = K$, $\mathcal{F}_\mathbb{N} = \mathcal{F}$, $\mathcal{F}'_\alpha = \mathcal{F}_{\mathbb{N} \setminus \alpha}$
Recall that $(F, \partial, r)$ was called ordinary if $\ker(\partial) = K$. Now call a Rota-Baxter algebra $(F, P)$ ordinary if $P$ is injective and $\text{im}(P) \cup K = F$. Then one can expand to canonical $(F, d, P)$.

Lemma Let $(F, P)$ be an ordinary Rota-Baxter algebra over $K$. Then $x \mapsto P(1)$ defines an embedding $(K[\![x]\!]$, $r x \mapsto 0)$ $\hookrightarrow (F, P)$ of Rota-Baxter algebras.
Recall that \((\mathcal{F}, \partial, \int)\) was called ordinary if \(\ker(\partial) = K\).
Ordinary Rota-Baxter Algebras

Recall that \((\mathcal{F}, \partial, \int)\) was called ordinary if \(\ker(\partial) = K\).
Now call a Rota-Baxter algebra \((\mathcal{F}, P)\) **ordinary**
- if \(P\) is injective
Recall that \((F, \partial, \int)\) was called ordinary if \(\ker(\partial) = K\).

Now call a Rota-Baxter algebra \((F, P)\) ordinary

- if \(P\) is injective
- and \(\text{im}(P) + K = F\).
Ordinary Rota-Baxter Algebras

Recall that \((\mathcal{F}, \partial, \int)\) was called ordinary if \(\ker(\partial) = K\).

Now call a Rota-Baxter algebra \((\mathcal{F}, P)\) **ordinary**

- if \(P\) is injective
- and \(\text{im}(P) + K = \mathcal{F}\).

Then one can expand to canonical \((\mathcal{F}, d, P)\).
Ordinary Rota-Baxter Algebras

Recall that $(\mathcal{F}, \partial, \int)$ was called ordinary if $\ker(\partial) = K$.

Now call a Rota-Baxter algebra $(\mathcal{F}, P)$ \textbf{ordinary}

- if $P$ is injective
- and $\text{im}(P) + K = \mathcal{F}$.

Then one can expand to canonical $(\mathcal{F}, d, P)$.

\textbf{Lemma}

Let $(\mathcal{F}, P)$ be an ordinary Rota-Baxter algebra over $K$. Then $x \mapsto P(1)$ defines an embedding $(K[x], \int_0^x) \hookrightarrow (\mathcal{F}, P)$ of Rota-Baxter algebras.
Hierarchical Rota-Baxter Algebras

Definition

A hierarchical Rota-Baxter algebra \((\mathcal{F}_n, \int^x_n)_{n \in \mathbb{N}}\) consists of a substitutive \(K\)-algebra \((\mathcal{F}_n)\) and commuting Rota-Baxter operators \(\int^x_n\) that satisfy the following axioms:

1. We have \(\int^x_n \mathcal{F}_m \subseteq \mathcal{F}_m\) and \(\int^x_n \mathcal{M}^*_m = \mathcal{M}^*_m \int^x_n\) for \(n \leq m\).
A hierarchical Rota-Baxter algebra \((F_n, \int^{x_n})_{n \in \mathbb{N}}\) consists of a substitutive \(K\)-algebra \((F_n)\) and commuting Rota-Baxter operators \(\int^{x_n}\) that satisfy the following axioms:

1. We have \(\int^{x_n} F_m \subseteq F_m\) and \(\int^{x_n} \tilde{M}^*_m = \tilde{M}^*_m \int^{x_n}\) for \(n \leq m\).
2. Every \((F_n, \int^{x_n})\) is an ordinary Rota-Baxter algebra over \(F_{n-1}\).
A hierarchical Rota-Baxter algebra \((\mathcal{F}_n, \int^x)_{n \in \mathbb{N}}\) consists of a substitutive \(K\)-algebra \((\mathcal{F}_n)\) and commuting Rota-Baxter operators \(\int^x\) that satisfy the following axioms:

1. We have \(\int^x \mathcal{F}_m \subseteq \mathcal{F}_m\) and \(\int^x \tilde{\mathcal{M}}_m^\ast = \tilde{\mathcal{M}}_m^\ast \int^x\) for \(n \leq m\).
2. Every \((\mathcal{F}_n, \int^x)\) is an ordinary Rota-Baxter algebra over \(\mathcal{F}_{n-1}\).
3. We have \(\tau^\ast \int^x = \int^x \tau^\ast\) for the transposition \(\tau = (i \ j)\).
Hierarchical Rota-Baxter Algebras

Definition

A **hierarchical Rota-Baxter algebra** \((\mathcal{F}_n, \int x^n)_{n \in \mathbb{N}}\) consists of a substitutive \(K\)-algebra \((\mathcal{F}_n)\) and commuting Rota-Baxter operators \(\int x^n\) that satisfy the following axioms:

1. We have \(\int x^n \mathcal{F}_m \subseteq \mathcal{F}_m\) and \(\int x^n \tilde{M}_m^* = \tilde{M}_m^* \int x^n\) for \(n \leq m\).
2. Every \((\mathcal{F}_n, \int x^n)\) is an ordinary Rota-Baxter algebra over \(\mathcal{F}_{n-1}\).
3. We have \(\tau^* \int x^i = \int x^j \tau^*\) for the transposition \(\tau = (i \ j)\).
4. The three substitution rules are satisfied (notation as before):

\[
\begin{align*}
\int x^\lambda^* &= \lambda^{-1} \lambda^* \int x^x \\
\int x^x T_x(e_i)^* &= (1 - E_x^*) T_x(e_i)^* \int x^x \\
\int x^x g L_x(e_{j-1} + v)^* \int x^x &= L_j^{-1}(v')^* (I_n \oplus e_j)^* \left( L_x(e_{j-1})^* \int x^x - \int x^x L_x(e_{j-1})^* \right) \int x^j \bar{g} L_j(v')^*
\end{align*}
\]
A hierarchical Rota-Baxter algebra \((F_n, \int x^n)_{n \in \mathbb{N}}\) consists of a substitutive \(K\)-algebra \((F_n)\) and commuting Rota-Baxter operators \(\int x^n\) that satisfy the following axioms:

1. We have \(\int x^n F_m \subseteq F_m\) and \(\int x^n \tilde{M}^* = \tilde{M}^* \int x^n\) for \(n \leq m\).
2. Every \((F_n, \int x^n)\) is an ordinary Rota-Baxter algebra over \(F_{n-1}\).
3. We have \(\tau^* \int x^i = \int x^j \tau^*\) for the transposition \(\tau = (i \ j)\).
4. The three substitution rules are satisfied (notation as before):

\[
\begin{align*}
\int x \lambda^* &= \lambda^{-1} \lambda^* \int x \\
\int x T_x(e_i)^* &= (1 - E_x^*) T_x(e_i)^* \int x \\
\int x g L_x(e_j-1 + v)^* \int x &= L_j^{-1}(v')^* (I_n \oplus e_j)^* \left( L_x(e_j-1)^* \int x - \int x L_x(e_j-1)^* \right) \int x^j \bar{g} L_j(v')^*
\end{align*}
\]

Crucial example: \(C^\infty(\mathbb{R}^\infty)\)
A hierarchical Rota-Baxter algebra \((\mathcal{F}_n, \int x^n)_{n \in \mathbb{N}}\) consists of a substitutive \(K\)-algebra \((\mathcal{F}_n)\) and commuting Rota-Baxter operators \(\int x^n\) that satisfy the following axioms:

1. We have \(\int x^n \mathcal{F}_m \subseteq \mathcal{F}_m\) and \(\int x^n \tilde{M}_m^* = \tilde{M}_m^* \int x^n\) for \(n \leq m\).
2. Every \((\mathcal{F}_n, \int x^n)\) is an ordinary Rota-Baxter algebra over \(\mathcal{F}_{n-1}\).
3. We have \(\tau^* \int x^i = \int x^j \tau^*\) for the transposition \(\tau = (i \ j)\).
4. The three substitution rules are satisfied (notation as before):

\[
\begin{align*}
\int x^\lambda^* &= \lambda^{-1} \lambda^* \int x \\
\int x^T_x(e_i)^* &= (1 - E_x^*) T_x(e_i)^* \int x \\
\int x^g L_x(e_{j-1} + v)^* \int x &= L_j^{-1}(v')^* (I_n \oplus e_j)^* \left( L_x(e_{j-1})^* \int x - \int x L_x(e_{j-1})^* \right) \int x^j \tilde{g} L_j(v')^*
\end{align*}
\]

Crucial example: \(C^\infty(\mathbb{R}^\infty)\)

\(\rightarrow\) Some subalgebras: \(C^\omega(\mathbb{R}^\infty)\), holonomies, \(K[x_1, x_2, \ldots]\)
Definition

A hierarchical Rota-Baxter algebra \((\mathcal{F}_n, \int^x n)_{n \in \mathbb{N}}\) consists of a substitutive \(K\)-algebra \((\mathcal{F}_n)\) and commuting Rota-Baxter operators \(\int^x n\) that satisfy the following axioms:

1. We have \(\int^x n \mathcal{F}_m \subseteq \mathcal{F}_m\) and \(\int^x n \tilde{\mathcal{M}}_m^* = \tilde{\mathcal{M}}_m^* \int^x n\) for \(n \leq m\).
2. Every \((\mathcal{F}_n, \int^x n)\) is an ordinary Rota-Baxter algebra over \(\mathcal{F}_{n-1}\).
3. We have \(\tau^* \int^x i = \int^x j \tau^*\) for the transposition \(\tau = (i \ j)\).
4. The three substitution rules are satisfied (notation as before):

\[
\begin{align*}
\int^x \lambda^* &= \lambda^{-1} \lambda^* \int^x \\
\int^x T_x(e_i)^* &= (1 - E_x^*) T_x(e_i)^* \int^x \\
\int^x g L_x(e_j - 1 + v)^* \int^x &= L_j^{-1}(v')^* (I_n \oplus e_j)^* \left( L_x(e_{j-1})^* \int^x - \int^x L_x(e_{j-1})^* \right) \int^x \bar{g} L_j(v')^*
\end{align*}
\]

Crucial example: \(C^\infty(\mathbb{R}^\infty)\)

\(\rightarrow\) Some subalgebras: \(C^\omega(\mathbb{R}^\infty)\), holonomics, \(K[x_1, x_2, \ldots]\)

\(\rightarrow\) Exponential polynomials \(K[x_1, x_2, \ldots, e^{\lambda x_1}, e^{\lambda x_2}, \ldots | \lambda \in K]\)
Verification of the Horizontal and Vertical Rule
Verification of the Horizontal and Vertical Rule

\[
\int_{x_1}^{x_1} T^* f(x_1, x_2, x_3, \ldots) = \int_{x_0}^{x_1} f(\xi + x_j, x_2, x_3, \ldots) \, d\xi = \int_{x_j}^{x_1+x_j} f(\xi, x_2, x_3, \ldots) \, d\xi
\]

\[
= \int_{0}^{x_1+x_j} f(\xi, x_2, x_3, \ldots) \, d\xi - \int_{0}^{x_j} f(\xi, x_2, x_3, \ldots) \, d\xi
\]

\[
= (1 - E_x^*) T^* \int_{x_1}^{x_1} f(x_1, x_2, x_3, \ldots)
\]
Verification of the Horizontal and Vertical Rule

\[
\int_{x_1}^{x_1} T^* f(x_1, x_2, x_3, \ldots) = \int_{0}^{x_1} f(\xi + x_j, x_2, x_3, \ldots) \, d\xi = \int_{x_j}^{x_1 + x_j} f(\bar{\xi}, x_2, x_3, \ldots) \, d\bar{\xi}
\]

\[
= \int_{0}^{x_1 + x_j} f(\xi, x_2, x_3, \ldots) \, d\xi - \int_{0}^{x_j} f(\xi, x_2, x_3, \ldots) \, d\xi
\]

\[
= (1 - E_x^*) T^* \int_{x_1}^{x_1} f(x_1, x_2, x_3, \ldots)
\]

\[
L_j(v')^* \int_{x_1}^{x_1} g(x_1) \, L_x(e_{j-1} + v)^* \int_{x_1}^{x_1} f(x_1, \ldots, x_n)
\]

\[
= L_j(v')^* \int_{0}^{x_1} g(\eta) \int_{0}^{\eta} f(\xi, x_{2j-1}, x_j + \eta, x_j+1n + v_{j+1}n \eta) \, d\xi \, d\eta
\]

\[
= L_j(v')^* \int_{0}^{x_1} \int_{\xi+x_j}^{x_1+x_j} g(\bar{\eta} - x_j) f(\xi, x_{2j-1}, \bar{\eta}, x_{j+1}n + v_{j+1}n (\bar{\eta} - x_j)) \, d\bar{\eta} \, d\xi
\]

\[
= \int_{0}^{x_1} \int_{\xi+x_j}^{x_1+x_j} \bar{g}(\eta, x_j) f(\xi, x_{2j-1}, \eta, x_{j+1}n + v_{j+1}n \eta) \, d\eta \, d\xi
\]

\[
= \int_{0}^{x_1} \int_{\xi+x_j}^{x_1+x_j} \ldots \, d\eta \, d\xi - \int_{0}^{x_1} \int_{0}^{\xi+x_j} \ldots \, d\eta \, d\xi
\]
For any \( \alpha = (\alpha_1, \ldots, \alpha_k) \), there is an embedding
\[
\iota_\alpha : K[X_{\alpha_1}, \ldots, X_{\alpha_k}] \hookrightarrow \mathcal{F}_\alpha
\]
\[
X_{\alpha_j} \mapsto x_{\alpha_j} := \int x_{\alpha_j} 1,
\]
and we have \( \pi^* p(x_{\alpha_1}, \ldots, x_{\alpha_k}) = p(x_{\pi(\alpha_1)}, \ldots, x_{\pi(\alpha_k)}) \) for all permutations \( \pi \) of \( (\alpha_1, \ldots, \alpha_k) \).
For any $\alpha = (\alpha_1, \ldots, \alpha_k)$, there is an embedding

$$\iota_\alpha : K[X_{\alpha_1}, \ldots, X_{\alpha_k}] \hookrightarrow F_\alpha$$

$$X_{\alpha_j} \mapsto x_{\alpha_j} := \int x_{\alpha_j} 1,$$

and we have $\pi^* p(x_{\alpha_1}, \ldots, x_{\alpha_k}) = p(x_{\pi(\alpha_1)}, \ldots, x_{\pi(\alpha_k)})$ for all permutations $\pi$ of $(\alpha_1, \ldots, \alpha_k)$.

For $\pi \in S_n$ and $i \leq n$ we have $\pi^* \int^{x_i} = \int^{x_j} \pi^*$ with $j := \pi(j)$. In particular, all $\int^{x_i} : F(i) \to F(i)$ are conjugates of $\int^{x_1} : F_1 \to F_1$ and hence ordinary Rota-Baxter operators.
For any $\alpha = (\alpha_1, \ldots, \alpha_k)$, there is an embedding

$$\iota_\alpha : K[X_{\alpha_1}, \ldots, X_{\alpha_k}] \hookrightarrow \mathcal{F}_\alpha$$

$$X_{\alpha_j} \mapsto x_{\alpha_j} := \int^{x_{\alpha_j}} 1,$$

and we have $\pi^*p(x_{\alpha_1}, \ldots, x_{\alpha_k}) = p(x_{\pi(\alpha_1)}, \ldots, x_{\pi(\alpha_k)})$ for all permutations $\pi$ of $(\alpha_1, \ldots, \alpha_k)$.

For $\pi \in S_n$ and $i \leq n$ we have $\pi^*\int^{x_i} = \int^{x_j} \pi^*$ with $j := \pi(j)$. In particular, all $\int^{x_i} : \mathcal{F}(i) \rightarrow \mathcal{F}(i)$ are conjugates of $\int^{x_1} : \mathcal{F}_1 \rightarrow \mathcal{F}_1$ and hence ordinary Rota-Baxter operators.

We have $\int^{x_n} cf = c \int^{x_n} f$ for all $c \in \mathcal{F}'(n)$ and $f \in \mathcal{F}$. In particular, $\int^{x_n} c = cx_n$. 
Simple Properties

1. For any $\alpha = (\alpha_1, \ldots, \alpha_k)$, there is an embedding

$$\iota_\alpha : K[X_{\alpha_1}, \ldots, X_{\alpha_k}] \hookrightarrow \mathcal{F}_\alpha$$

$$X_{\alpha_j} \mapsto x_{\alpha_j} := \int x_{\alpha_j} 1,$$

and we have $\pi^* p(x_{\alpha_1}, \ldots, x_{\alpha_k}) = p(x_{\pi(\alpha_1)}, \ldots, x_{\pi(\alpha_k)})$ for all permutations $\pi$ of $(\alpha_1, \ldots, \alpha_k)$.

2. For $\pi \in S_n$ and $i \leq n$ we have $\pi^* \int x_i = \int x_j \pi^*$ with $j := \pi(j)$. In particular, all $\int x_i : \mathcal{F}_{(i)} \rightarrow \mathcal{F}_{(i)}$ are conjugates of $\int x_1 : \mathcal{F}_1 \rightarrow \mathcal{F}_1$ and hence ordinary Rota-Baxter operators.

3. We have $\int x^n c f = c \int x^n f$ for all $c \in \mathcal{F}'_{(n)}$ and $f \in \mathcal{F}$. In particular, $\int x^n c = cx_n$.

4. The embedding $\iota_\alpha$ of Item (1) is a homomorphism of Rota-Baxter algebras in the sense that $\iota_\alpha \circ \int_0 X_{\alpha_j} = \int x_{\alpha_j} \circ \iota_\alpha$ for $j = 1, \ldots, k$. 
Simple Properties

For any \( \alpha = (\alpha_1, \ldots, \alpha_k) \), there is an embedding

\[
\iota_\alpha : K[X_{\alpha_1}, \ldots, X_{\alpha_k}] \hookrightarrow \mathcal{F}_\alpha
\]

\[
X_{\alpha_j} \mapsto x_{\alpha_j} := \int^{x_{\alpha_j}} 1,
\]

and we have \( \pi^* p(x_{\alpha_1}, \ldots, x_{\alpha_k}) = p(x_{\pi(\alpha_1)}, \ldots, x_{\pi(\alpha_k)}) \) for all permutations \( \pi \) of \( (\alpha_1, \ldots, \alpha_k) \).

For \( \pi \in S_n \) and \( i \leq n \) we have \( \pi^* \int^{x_i} = \int^{x_j} \pi^* \) with \( j := \pi(j) \). In particular, all \( \int^{x_i} : \mathcal{F}(i) \to \mathcal{F}(i) \) are conjugates of \( \int^{x_1} : \mathcal{F}_1 \to \mathcal{F}_1 \) and hence ordinary Rota-Baxter operators.

We have \( \int^{x_n} cf = c \int^{x_n} f \) for all \( c \in \mathcal{F}'_n \) and \( f \in \mathcal{F} \). In particular, \( \int^{x_n} c = cx_n \).

The embedding \( \iota_\alpha \) of Item (1) is a homomorphism of Rota-Baxter algebras in the sense that \( \iota_\alpha \circ \int_0^{X_{\alpha_j}} = \int^{x_{\alpha_j}} \circ \iota_\alpha \) for \( j = 1, \ldots, k \).

If \( M \in \mathcal{M}(K) \) vanishes in the \( i \)-th column, then \( M^*(\mathcal{F}) \subset \mathcal{F}'_i \).
1. For any \( \alpha = (\alpha_1, \ldots, \alpha_k) \), there is an embedding
\[
\iota_\alpha : K[X_{\alpha_1}, \ldots, X_{\alpha_k}] \hookrightarrow \mathcal{F}_\alpha
\]
\[
X_{\alpha_j} \mapsto x_{\alpha_j} := \int x_{\alpha_j} 1,
\]
and we have \( \pi^* p(x_{\alpha_1}, \ldots, x_{\alpha_k}) = p(x_{\pi(\alpha_1)}, \ldots, x_{\pi(\alpha_k)}) \) for all permutations \( \pi \) of \( (\alpha_1, \ldots, \alpha_k) \).

2. For \( \pi \in S_n \) and \( i \leq n \) we have \( \pi^* \int x^i = \int x^j \pi^* \) with \( j := \pi(j) \). In particular, all \( \int x^i : \mathcal{F}(i) \to \mathcal{F}(i) \) are conjugates of \( \int x^1 : \mathcal{F}_1 \to \mathcal{F}_1 \) and hence ordinary Rota-Baxter operators.

3. We have \( \int x^n c f = c \int x^n f \) for all \( c \in \mathcal{F}'(n) \) and \( f \in \mathcal{F} \). In particular, \( \int x^n c = cx_n \).

4. The embedding \( \iota_\alpha \) of Item (1) is a homomorphism of Rota-Baxter algebras in the sense that \( \iota_\alpha \circ \int_0^X x_{\alpha_j} = \int x_{\alpha_j} \circ \iota_\alpha \) for \( j = 1, \ldots, k \).

5. If \( M \in \mathcal{M}(K) \) vanishes in the \( i \)-th column, then \( M^*(\mathcal{F}) \subset \mathcal{F}'(i) \).

6. We have \( E_i^* \int x^i = 0 \) for all \( i > 0 \).
Admissible Coefficient Algebras

Induced hierarchy of ordinary $(F_1, r)$:

- Ascending algebra $(G_n, r_x)$ for $n \in \mathbb{N}$

Algebras $G_n := G \otimes \mathbb{K}$ with $f_1 \otimes \cdots \otimes f_n \mapsto f_1 \otimes \cdots \otimes f_n \otimes 1$

Rota-Baxter operators $r_x := 1 \otimes (n - 1) \otimes r$.

Definition: Let $(F_n, r_x)$ be a hierarchical Rota-Baxter algebra over a field $\mathbb{K}$.

A substitutive ordinary integro-differential algebra $(G_1, r)$ over $\mathbb{K}$ is called an admissible coefficient domain if its induced hierarchy $(G_n, r_x)$ for $n \in \mathbb{N}$ is a hierarchical integro-differential subalgebra of $(F_n, r_x)$.

Minimal example: $\mathbb{K}[x] = \mathbb{K}[x_1, x_2, \ldots]$ for any $(F, r)$.

Important for applications: $\mathbb{K}[x, e] \subset \mathcal{C}_\infty(R)$.
Admissible Coefficient Algebras

Induced hierarchy of ordinary \((\mathcal{F}_1, \int)\):

\[
\text{Ascending algebra } (G_n, r_n)_{n \in \mathbb{N}}
\]

Algebras \(G_n := G \otimes n\) with \(f_1 \otimes \cdots \otimes f_n \mapsto f_1 \otimes \cdots \otimes f_n \otimes 1\)

Rota-Baxter operators \(r_n \mapsto 1 \otimes (n - 1) \otimes r_n\)

Definition

Let \((\mathcal{F}_n, r_n)_{n \in \mathbb{N}}\) be a hierarchical Rota-Baxter algebra over a field \(K\).

A substitutive ordinary integro-differential algebra \((G_1, r)\) over \(K\) is called an admissible coefficient domain if its induced hierarchy \((G_n, r_n)_{n \in \mathbb{N}}\) is a hierarchical integro-differential subalgebra of \((\mathcal{F}_n, r_n)_{n \in \mathbb{N}}\).

Minimal example

\(K[x] = K[x_1, x_2, \ldots]\) for any \((\mathcal{F}, r)\)

Important for applications:

\(K[x, e Kx] \subset C_\infty (R_\infty)\)
Admissible Coefficient Algebras

Induced hierarchy of ordinary \((\mathcal{F}_1, \int)\):

- Ascending algebra \((\mathcal{G}_n, \int x^n)_{n \in \mathbb{N}}\)
Admissible Coefficient Algebras

Induced hierarchy of ordinary \((F_1, \int)\):

- Ascending algebra \((G_n, \int^{x_n})_{n \in \mathbb{N}}\)
- Algebras \(G_n := G^\otimes n\) with \(f_1 \otimes \cdots \otimes f_n \mapsto f_1 \otimes \cdots \otimes f_n \otimes 1\)
Admissible Coefficient Algebras

Induced hierarchy of ordinary \((\mathcal{F}_1, \int)\):

- Ascending algebra \((\mathcal{G}_n, \int^x_n)_{n \in \mathbb{N}}\)
- Algebras \(\mathcal{G}_n := \mathcal{G} \otimes^n\) with \(f_1 \otimes \cdots \otimes f_n \mapsto f_1 \otimes \cdots \otimes f_n \otimes 1\)
- Rota-Baxter operators \(\int^x_n := 1 \otimes (n-1) \otimes \int\)
Admissible Coefficient Algebras

Induced hierarchy of ordinary \((\mathcal{F}_1, \int)\):

- Ascending algebra \((\mathcal{G}_n, \int^{x_n})_{n \in \mathbb{N}}\)
- Algebras \(\mathcal{G}_n := \mathcal{G} \otimes^n\) with \(f_1 \otimes \cdots \otimes f_n \mapsto f_1 \otimes \cdots \otimes f_n \otimes 1\)
- Rota-Baxter operators \(\int^{x_n} := 1 \otimes (n-1) \otimes \int\)

**Definition**

Let \((\mathcal{F}_n, \int^{x_n})_{n \in \mathbb{N}}\) be a hierarchical Rota-Baxter algebra over a field \(K\). A substitutive ordinary integro-differential algebra \((\mathcal{G}_1, \int)\) over \(K\) is called an **admissible coefficient domain** if its induced hierarchy \((\mathcal{G}_n, \int^{x_n})_{n \in \mathbb{N}}\) is a hierarchical integro-differential subalgebra of \((\mathcal{F}_n, \int^{x_n})_{n \in \mathbb{N}}\).
Admissible Coefficient Algebras

Induced hierarchy of ordinary \((\mathcal{F}_1, \int)\):

- Ascending algebra \((\mathcal{G}_n, \int^{x_n})_{n \in \mathbb{N}}\)
- Algebras \(\mathcal{G}_n := \mathcal{G} \otimes^n\) with \(f_1 \otimes \cdots \otimes f_n \mapsto f_1 \otimes \cdots \otimes f_n \otimes 1\)
- Rota-Baxter operators \(\int^{x_n} := 1 \otimes (n-1) \otimes \int\)

**Definition**

Let \((\mathcal{F}_n, \int^{x_n})_{n \in \mathbb{N}}\) be a hierarchical Rota-Baxter algebra over a field \(K\). A substitutive ordinary integro-differential algebra \((\mathcal{G}_1, \int)\) over \(K\) is called an **admissible coefficient domain** if its induced hierarchy \((\mathcal{G}_n, \int^{x_n})_{n \in \mathbb{N}}\) is a hierarchical integro-differential subalgebra of \((\mathcal{F}_n, \int^{x_n})_{n \in \mathbb{N}}\).

Minimal example \(K[x] = K[x_1, x_2, \ldots]\) for any \((\mathcal{F}, \int)\)
Admissible Coefficient Algebras

Induced hierarchy of ordinary \((\mathcal{F}_1, \int)\):
- Ascending algebra \((\mathcal{G}_n, \int^{x_n})_{n \in \mathbb{N}}\)
- Algebras \(\mathcal{G}_n := \mathcal{G} \otimes^n\) with \(f_1 \otimes \cdots \otimes f_n \mapsto f_1 \otimes \cdots \otimes f_n \otimes 1\)
- Rota-Baxter operators \(\int^{x_n} := 1 \otimes (n-1) \otimes \int\)

**Definition**

Let \((\mathcal{F}_n, \int^{x_n})_{n \in \mathbb{N}}\) be a hierarchical Rota-Baxter algebra over a field \(K\). A substitutive ordinary integro-differential algebra \((\mathcal{G}_1, \int)\) over \(K\) is called an *admissible coefficient domain* if its induced hierarchy \((\mathcal{G}_n, \int^{x_n})_{n \in \mathbb{N}}\) is a hierarchical integro-differential subalgebra of \((\mathcal{F}_n, \int^{x_n})_{n \in \mathbb{N}}\).

Minimal example \(K[x] = K[x_1, x_2, \ldots]\) for any \((\mathcal{F}, \int)\)
Important for applications: \(K[x, e^{Kx}] \subset C^\infty(\mathbb{R}^\infty)\)
Can expand every $g \in \mathcal{G}$ as

$$g = \sum_{k=1}^{r} g_{k,1} \cdots g_{k,n}$$

with $g_{k,i} \in \mathcal{G}(i)$ for $i \in \{1, \ldots, n\}$. 
Can expand every \( g \in \mathcal{G} \) as
\[
g = \sum_{k=1}^{r} g_{k,1} \cdots g_{k,n}
\]
with \( g_{k,i} \in \mathcal{G}(i) \) for \( i \in \{1, \ldots, n\} \).

Use some kind of Sweedler notation:
Coalgebra Structure for Coefficients

Can expand every \( g \in G \) as

\[
g = \sum_{k=1}^{r} g_{k,1} \cdots g_{k,n}
\]

with \( g_{k,i} \in G_{(i)} \) for \( i \in \{1, \ldots, n\} \).

Use some kind of **Sweedler notation**:
- Abbreviate the \( g_{1,i}, g_{2,i}, \ldots \in G_{(i)} \) by \( g(i) \) with implied summation.
Coalgebra Structure for Coefficients

Can expand every $g \in G$ as

$$g = \sum_{k=1}^{r} g_{k,1} \cdots g_{k,n}$$

with $g_{k,i} \in G_{(i)}$ for $i \in \{1, \ldots, n\}$.

Use some kind of **Sweedler notation**:
- Abbreviate the $g_{1,i}, g_{2,i}, \ldots \in G_{(i)}$ by $g_{(i)}$ with implied summation.
- Hence expansion is $g = g_{(1)} \cdots g_{(n)}$. 
Can expand every \( g \in \mathcal{G} \) as

\[
g = \sum_{k=1}^{r} g_{k,1} \cdots g_{k,n}
\]

with \( g_{k,i} \in \mathcal{G}_{(i)} \) for \( i \in \{1, \ldots, n\} \).

Use some kind of Sweedler notation:

- Abbreviate the \( g_{1,i}, g_{2,i}, \ldots \in \mathcal{G}_{(i)} \) by \( g(i) \) with implied summation.
- Hence expansion is \( g = g(1) \cdots g(n) \).
- More generally, \( g_{k,(\alpha)} := g_{k,\alpha_1} \cdots g_{k,\alpha_r} \) so that \( g = g(1)g(1)' \) etc.
Can expand every \( g \in \mathcal{G} \) as
\[
g = \sum_{k=1}^{r} g_{k,1} \cdots g_{k,n}
\]
with \( g_{k,i} \in \mathcal{G}_{(i)} \) for \( i \in \{1, \ldots, n\} \).

Use some kind of **Sweedler notation:**

- Abbreviate the \( g_{1,i}, g_{2,i}, \ldots \in \mathcal{G}_{(i)} \) by \( g_{(i)} \) with implied summation.
- Hence expansion is \( g = g_{(1)} \cdots g_{(n)} \).
- More generally, \( g_{k,\alpha} := g_{k,\alpha_1} \cdots g_{k,\alpha_r} \) so that \( g = g_{(1)}g_{(1)'} \) etc.
- Abbreviate shifted factors by \( (i \ j)^* g_{1,i}, (i \ j)^* g_{2,i}, \ldots \in \mathcal{G}_{(j)} \) by \( g_{(i:j)} \).
Can expand every $g \in G$ as

$$g = \sum_{k=1}^{r} g_{k,1} \cdots g_{k,n}$$

with $g_{k,i} \in G_i$ for $i \in \{1, \ldots, n\}$.

Use some kind of **Sweedler notation**:

- Abbreviate the $g_{1,i}, g_{2,i}, \ldots \in G_i$ by $g(i)$ with implied summation.
- Hence expansion is $g = g(1) \cdots g(n)$.
- More generally, $g_{k,\alpha} := g_{k,\alpha_1} \cdots g_{k,\alpha_r}$ so that $g = g(1) g(1)'$ etc.
- Abbreviate shifted factors by $(i \, j)^* g_{1,i}, (i \, j)^* g_{2,i}, \ldots \in G(j)$ by $g(i:j)$.
- Similarly, $(i \, j)^* g_{1,(i)'}, (i \, j)^* g_{1,(i)'}, \ldots \in G(j)'$ written as $g(i:j)'$.
Normalization of General Line Integrators

Proposition

Let \((\mathcal{F}_n, \int^{x_n})_{n \in \mathbb{N}}\) be a hierarchical Rota-Baxter algebra over a field \(K\), and let \(\mathcal{G}_1\) be an admissible coefficient domain for \(\mathcal{F}\). Then for \(M \in \mathcal{M}_n\) and \(g \in \mathcal{G}_1\) with \(g(x_j) := (1 \, j)^* g\) and \(j \in \{1, \ldots, n\}\) we have

\[
\int^{x_j} g(x_j) M^* = \begin{cases} 
M^{-1}_{ij} \tilde{g}(1:j)' (1 - E^*) \tilde{M}^* \int^{x_i} \hat{M}_{ij}^* (\tilde{g}(1:i)) L_i(l)^* & \text{if } i \neq \infty, \\
(\int^{x_j} g(x_j)) M^* & \text{othw.}
\end{cases}
\]

By definition \(i = \min\{k \mid M_{kj} \neq 0\}\), with \(\tilde{M} \in \mathcal{M}_n\) and \(l \in K_{n-i}\) by one sweep of Gaussian elimination if the minimum exists, and by convention \(i = \infty\) otherwise. Moreover, \(\tilde{g} = M_{i \bullet}^* g\) and \(\hat{M}_{ij} = d_{i,1/M_{ij}}\).
Proposition

Let \((\mathcal{F}_n, \int^{x_n})_{n \in \mathbb{N}}\) be a hierarchical Rota-Baxter algebra over a field \(K\), and let \(\mathcal{G}_1\) be an admissible coefficient domain for \(\mathcal{F}\). Then for \(i < j\) and arbitrary vectors \(v \in K^{n-i}\), \(w \in K^{n-j}\) and functions \(g, h \in \mathcal{G}_1\) with \(g(x_i) := (1^i) g\) and \(h(x_j) := (1^j) h\) we have

\[
\int^{x_j} h(x_j) L_j(w) \int^{x_i} g(x_i) L_i(v) = (1 - E_j^*) \int^{x_i} g(x_i) L_i(v') \int^{x_j} h(x_j) L_j(w)
\]

with \(v' = L_{j-i}^{-1}(w) v \in K^{n-i}\) as earlier.
Proposition

Let \((\mathcal{F}_n, \int^{x_n})_{n \in \mathbb{N}}\) be a hierarchical Rota-Baxter algebra over a field \(K\), and let \(\mathcal{G}_1\) be an admissible coefficient domain for \(\mathcal{F}\). Then for \(i \in \mathbb{N}\) and arbitrary vectors \(v, w \in K^{n-i}\) with \(w \neq 0\) we have

\[
w_k \int^{x_i} h(x_i) L_i(w) \ast \int^{x_i} g(x_i) L_i(v)^* = L_k^{-1}(w')^* \sigma^*(\bar{h}(n+1)) \times \]
\[
\times \left( L_i(\bar{w})^* \int^{x_i} \tilde{h}_{(1:k)}^* L_i(v')^* - \int^{x_i} \tilde{h}_{(1:k)}^* L_i(v' + \bar{w})^* \right) \int^{x_k} \tilde{h}_{(1:k)} L_k(w')^*
\]

where \(\bar{h} := (e_k/w_k - e_{n+1}/w_k)^* h = h(k)h(n+1) \in \mathcal{F}_{(k,n+1)}\) with slack transposition \(\sigma := (k \cdot n + 1)\), and \(\tilde{h} := L_i(v')_k^* (1 k)^* \tilde{h}(k) \in \mathcal{G}_{(i,k)}\).

The remaining notation is as earlier.
The Operator Ring

Definition

Let \(( G_1, r )\) be an ordinary integro-differential algebra over a field \( K \) with induced hierarchy \(( G_n, r_x n ) \ n \in \mathbb{N} \). Then the ring of partial integral operators over \( G \) is defined as the quotient of the \( K \)-algebra \( G[ r ] = G \oplus K \oplus K[ M ] \) \( \sim = \) with the congruence \( M \ast g \sim = ( M \cdot g ) \)

\[ M \ast g \sim = 0 \text{ if } M_i \cdot g = 0 \]

\[ A_j g( x_i ) \sim = g( x_i ) A_j A_i g( x_j ) \sim = g( x_j ) A_i \]

\[ M \ast \sim = M - 1_{ij} \tilde{g}(1:j) \times (1 - E^*_j) \tilde{M} \ast \]

\[ A_j g( x_j ) \sim = ( r_x j g( x_j ) ) A_j \sim = ( r_x j g( x_j ) ) \]

\[ A_j h( x_j ) \ast L_j( w ) \ast A_i g( x_i ) \ast L_i( v ) \ast \sim = (1 - E^*_j) A_i g( x_i ) L_i( v ') \ast A_j h( x_j ) L_j( w ) \ast A_i h( x_i ) L_i( w ') \ast \sim = w - 1_k L - 1_k (w') \ast \sigma^*(\bar{h}(n+1)) \times (L_i( \bar{w}) \ast A_i \tilde{h}(1:k) L_i( v ') - A_i \tilde{h}(1:k) L_i( v ' + \bar{w})) \ast A_k \tilde{h}(1:k) L_k( w ') \ast \]
The Operator Ring

Definition

Let \((G_1, \int)\) be an ordinary integro-differential algebra over a field \(K\) with induced hierarchy \((G_n, \int^{x_n})_{n \in \mathbb{N}}\). Then the ring of partial integral operators over \(G\) is defined as the quotient of the \(K\)-algebra

\[ G[\int] = G \amalg_K K[M]^* \amalg_K K[A] / \cong \]

with the congruence \(\cong\) given below.

\[
M^*g \cong (M \cdot g) M^* \quad \quad M^*A_i \cong 0 \quad \text{if} \quad M_\bullet = 0
\]

\[
A_j g(x_i) \cong g(x_i) A_j
\]

\[
A_j g(x_j) M^* \cong M_{ij}^{-1} \tilde{g}(1:j)' (1 - E_j^*) \tilde{M}^* A_i \tilde{M}_{ij}^* (\tilde{g}(1:i)) L_i(l)^*
\]

\[
A_j g(x_j) M^* \cong (\int^x g(x_j)) M^*
\]

\[
A_j h(x_j) L_j(w)^* A_i g(x_i) L_i(v)^* \cong (1 - E_j^*) A_i g(x_i) L_i(v')^* A_j h(x_j) L_j(w)^*
\]

\[
A_i h(x_i) L_i(w)^* A_i g(x_i) L_i(v)^* \cong w_k^{-1} L_k^{-1}(w')^* \sigma^* (\tilde{h}(n+1)) \times
\]

\[
\times \left( L_i(\bar{w})^* A_i \tilde{h}(1:k)' L_i(v')^* - A_i \tilde{h}(1:k)' L_i(v' + \bar{w})^* \right) A_k \tilde{h}(1:k) L_k(w')^*
\]

\[
A_j g(x_j) A_j \cong (\int^x g(x_j)) A_j - A_j (\int^x g(x_j))
\]
Natural Action and Termination

Proposition

Let \((F_n, r_x) \in N\) be a hierarchical Rota-Baxter algebra over a field \(K\), and let \(G_1\) be an admissible coefficient domain for \(F\). Then the natural action \(G[r] \times F \to F\) induced by \(g \cdot f = gf\), \(M^* \cdot f = M^*(f)\), and \(A_i \cdot f = r_x i f\) is well-defined. This follows from the propositions given above.

Now introduce a suitable term order on underlying word monoid.

Theorem

Let \((G_1, r)\) be an ordinary integro-differential algebra over a field \(K\). Orienting the rules of the Table from left to right, one obtains a Noetherian reduction system.
Natural Action and Termination

Proposition

Let \((\mathcal{F}_n, \int^{x_n})_{n \in \mathbb{N}}\) be a hierarchical Rota-Baxter algebra over a field \(K\), and let \(G_1\) be an admissible coefficient domain for \(\mathcal{F}\). Then the natural action \(G[\int] \times \mathcal{F} \rightarrow \mathcal{F}\) induced by \(g \cdot f = gf\), \(M^* \cdot f = M^*(f)\) and \(A_i \cdot f = \int^{x_i} f\) is well-defined.

This follows from the propositions given above.

Now introduce a suitable term order on underlying word monoid.

Theorem

Let \((G_1, r)\) be an ordinary integro-differential algebra over a field \(K\). Orienting the rules of the Table from left to right, one obtains a Noetherian reduction system.
Proposition

Let \((\mathcal{F}_n, \int^x f)_n \in \mathbb{N}\) be a hierarchical Rota-Baxter algebra over a field \(K\), and let \(G_1\) be an admissible coefficient domain for \(\mathcal{F}\). Then the natural action \(G[\int] \times \mathcal{F} \to \mathcal{F}\) induced by \(g \cdot f = gf\), \(M^* \cdot f = M^*(f)\) and \(A_i \cdot f = \int^{x_i} f\) is well-defined.

This follows from the propositions given above.
**Proposition**

Let \((\mathcal{F}_n, \int^x n)_{n \in \mathbb{N}}\) be a hierarchical Rota-Baxter algebra over a field \(K\), and let \(G_1\) be an admissible coefficient domain for \(\mathcal{F}\). Then the natural action \(G[\int] \times \mathcal{F} \to \mathcal{F}\) induced by \(g \cdot f = gf\), \(M^* \cdot f = M^*(f)\) and \(A_i \cdot f = \int^x_i f\) is well-defined.

This follows from the propositions given above.

Now introduce a suitable term order on underlying word monoid.
Proposition

Let \((F_n, \int^{x_n})_{n \in \mathbb{N}}\) be a hierarchical Rota-Baxter algebra over a field \(K\), and let \(G_1\) be an admissible coefficient domain for \(F\). Then the natural action \(G[\int] \times F \to F\) induced by \(g \cdot f = gf, M^* \cdot f = M^*(f)\) and \(A_i \cdot f = \int^{x_i} f\) is well-defined.

This follows from the propositions given above.

Now introduce a suitable term order on underlying word monoid.

Theorem

Let \((G_1, \int)\) be an ordinary integro-differential algebra over a field \(K\). Orienting the rules of the Table from left to right, one obtains a Noetherian reduction system.
Conjectured Canonical Forms

Conjectured Canonical Forms

Conjectured Canonical Forms
Conjectured Canonical Forms

- **Line integrator** of index $i$ is $A_i b(x_i) L_i(v)^*$ with $v \in K^{n-1}$ and a basis element $b \in G_1$.
Conjectured Canonical Forms

- **Line integrator** of index $i$ is $A_i b(x_i)L_i(v)^*$ with $v \in K^{n-1}$ and a basis element $b \in \mathcal{G}_1$.

- **Volume integrator** is a word of the form $b M^* J_1 \cdots J_r$ for line integrators $J_1, \ldots, J_r$ with indices $i_1 < \cdots < i_r$ and $M^* \in \mathcal{M}(K)^*$ with $M_{i_1} \neq 0$ if $r > 0$.

Easy to check: The volume integrators span $\mathcal{G}[r]$ over $K$.

Conjecture: They are linearly independent over $K$.

Then we have a system of canonical forms.
Conjectured Canonical Forms

- **Line integrator** of index $i$ is $A_i b(x_i) L_i(v)^*$ with $v \in K^{n-1}$ and a basis element $b \in G_1$.

- **Volume integrator** is a word of the form $b M^* J_1 \cdots J_r$ for line integrators $J_1, \ldots, J_r$ with indices $i_1 < \cdots < i_r$ and $M^* \in \mathcal{M}(K)^*$ with $M_{i_1} \neq 0$ if $r > 0$.

**Easy to check:** The volume integrators span $G[\int]$ over $K$. 
Conjectured Canonical Forms

- **Line integrator** of index $i$ is $A_i b(x_i) L_i(v)^* \text{ with } v \in K^{n-1}$ and a basis element $b \in G_1$.

- **Volume integrator** is a word of the form $b M^* J_1 \cdots J_r$ for line integrators $J_1, \ldots, J_r$ with indices $i_1 < \cdots < i_r$ and $M^* \in M(K)^*$ with $M_{i_1} \neq 0$ if $r > 0$.

**Easy to check**: The volume integrators span $G[\mathcal{J}]$ over $K$.

**Conjecture**: They are linearly independent over $K$. 
Conjecture: Canonical Forms

- **Line integrator** of index $i$ is $A_i b(x_i) L_i(v)^*$ with $v \in K^{n-1}$ and a basis element $b \in G_1$.

- **Volume integrator** is a word of the form $b M^* J_1 \cdots J_r$ for line integrators $J_1, \ldots, J_r$ with indices $i_1 < \cdots < i_r$ and $M^* \in M(K)^*$ with $M_{i_1} \neq 0$ if $r > 0$.

Easy to check: The volume integrators span $G[\int]$ over $K$.

Conjecture: They are linearly independent over $K$.

Then we have a system of canonical forms.
Additional Rules for Derivations

Assume \((F_n, x^n, \partial x^n)\) is hierarchical integro-differential algebra.

Add indeterminates \(D_n\) for action of \(\partial x^n\), impose the relations:

\[ D_i M^* = \sum_k M_{ik} M^* D_k \]
\[ D_i D_j = D_j D_i \]
\[ D_i f(x_i) = f(x_i) D_i + f'(x_i) D_i \]
\[ D_i A_i = 1 \]
\[ A_j D_i = A_j D_i \]
\[ L_i(v) \star D_i = (f(x_i) - A_i f'(x_i) - f_i(0) E_i) L_i(v) \star - \sum_{j>i} v_j A_i f(x_i) L_i(v) \star D_j \]

Canonical forms similar but with certain \(D_\alpha\) on the right.
Additional Rules for Derivations

Assume $(\mathcal{F}_n, \int^{x_n}, \partial_{x_n})$ is hierarchical integro-differential algebra.
Additional Rules for Derivations

Assume $(\mathcal{F}_n, \int^{x_n}, \partial_{x_n})$ is **hierarchical integro-differential algebra**.

Add indeterminates $D_n$ for action of $\partial_{x_n}$, impose the relations:
Additional Rules for Derivations

Assume \((\mathcal{F}_n, \int^{x_n}, \partial_{x_n})\) is hierarchical integro-differential algebra.

Add indeterminates \(D_n\) for action of \(\partial_{x_n}\), impose the relations:

\[
\begin{align*}
D_i M^* &= \sum_k M_{ik} M^* D_k & D_i D_j &= D_j D_i \\
D_i f(x_i) &= f(x_i) D_i + f'(x_i) & D_i f(x_j) &= f(x_j) D_i \\
D_i A_i &= 1 & D_i A_j &= A_j D_i \\
A_i f(x_i) L_i(v)^* D_i &= (f(x_i) - A_i f'_i(x_i) - f_i(0) E_i^*) L_i(v)^* - \sum_{j>i} v_j A_i f(x_i) L_i(v)^* D_j 
\end{align*}
\]
**Additional Rules for Derivations**

Assume \((\mathcal{F}_n, \int^x_n, \partial_{x_n})\) is **hierarchical integro-differential algebra**.

Add indeterminates \(D_n\) for action of \(\partial_{x_n}\), impose the relations:

<table>
<thead>
<tr>
<th>Equation</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_i M^* = \sum_k M_{ik} M^* D_k)</td>
<td>(D_i D_j = D_j D_i)</td>
</tr>
<tr>
<td>(D_i f(x_i) = f(x_i) D_i + f'(x_i))</td>
<td>(D_i f(x_j) = f(x_j) D_i)</td>
</tr>
<tr>
<td>(D_i A_i = 1)</td>
<td>(D_i A_j = A_j D_i)</td>
</tr>
<tr>
<td>(A_i f(x_i) L_i(v)^* D_i = (f(x_i) - A_i f'<em>i(x_i) - f_i(0) E_i^<em>) L_i(v)^</em> - \sum</em>{j&gt;i} v_j A_i f(x_i) L_i(v)^* D_j)</td>
<td></td>
</tr>
</tbody>
</table>

Canonical forms similar but with certain \(D^\alpha\) on the right.
LPDE Example Revisited

Cauchy problem:

$$u_{tt} - 4u_{tx} + 4u_{xx} - 9u_{yy} = f,$$

$$u(0, x, y) = f_1(x, y),$$
$$u_t(0, x, y) = f_2(x, y).$$

Signal and state operators:

$$Gf(t, x, y) = \int_0^t \int_0^\sigma r(t - 2\tau, y - 3\tau + 6\sigma) f(\tau, x + 2t - 2\tau) d\tau d\sigma.$$

$$H(f_1, f_2) = f_1(x + 2t, y - 3t) + \int_0^t r(t - 2\tau) \left[f_2(2D_x f_1 + 3D_y f_1)(x + 2t, y - 3t + 6\tau)\right] d\tau.$$

Factor problems:

$$u_t - 2u_x \pm 3u_y = f,$$

$$u(0, x, y) = f \pm (x, y).$$

$$H \pm f \pm (t, x, y) = f \pm (x + 2t, y \mp 3t)$$

$$G \pm f(t, x, y) = \int_0^t f(\tau, x + 2t - 2\tau, y \mp 3t \pm 3\tau) d\tau.$$
LPDE Example Revisited

Cauchy problem:

\[
\begin{align*}
    u_{tt} - 4u_{tx} + 4u_{xx} - 9u_{yy} &= f, \\
    u(0,x,y) &= f_1(x,y), \\
    u_t(0,x,y) &= f_2(x,y)
\end{align*}
\]
LPDE Example Revisited

Cauchy problem:

\[
\begin{align*}
    u_{tt} - 4u_{tx} + 4u_{xx} - 9u_{yy} &= f, \\
    u(0, x, y) &= f_1(x, y), \\
    u_t(0, x, y) &= f_2(x, y)
\end{align*}
\]

Signal and state operators:

\[
Gf(t, x, y) = \int_0^t \int_0^\sigma f(\tau, x + 2t - 2\tau, y - 3t - 3\tau + 6\sigma) \, d\tau \, d\sigma.
\]

\[
H(f_1, f_2) = f_1(x+2t, y-3t) + \int_0^t (f_2 - 2D_xf_1 + 3D_yf_1)(x+2t, y-3t+6\tau) \, d\tau
\]
LPDE Example Revisited

Cauchy problem:
\[
\begin{align*}
    u_{tt} - 4u_{tx} + 4u_{xx} - 9u_{yy} &= f, \\
    u(0, x, y) &= f_1(x, y), \\
    u_t(0, x, y) &= f_2(x, y)
\end{align*}
\]

Signal and state operators:
\[
\begin{align*}
    G f(t, x, y) &= \int_0^t \int_0^\sigma f(\tau, x + 2t - 2\tau, y - 3t - 3\tau + 6\sigma) \, d\tau \, d\sigma. \\
    H(f_1, f_2) &= f_1(x + 2t, y - 3t) + \int_0^t (f_2 - 2D_x f_1 + 3D_y f_1)(x + 2t, y - 3t + 6\tau) \, d\tau
\end{align*}
\]

Factor problems:
\[
\begin{align*}
    u_t - 2u_x \pm 3u_y &= f, \\
    u(0, x, y) &= f^\pm(x, y)
\end{align*}
\]
\[
\begin{align*}
    H^{\pm} f^\pm(t, x, y) &= f^\pm(x + 2t, y \mp 3t) \\
    G^{\pm} f(t, x, y) &= \int_0^t f(\tau, x + 2t - 2\tau, y \mp 3t \pm 3\tau) \, d\tau
\end{align*}
\]
Factorization Examples for LPDEs

Unbounded wave equation:  

\[(D_{tt} - D_{xx}, [L_t, L_tD_t, L_x, R_x]) = (D_{tt} - D_x, [L_t]) \cdot (D_{tt} + D_x, [L_t])\]

\[u_{tt} - u_{xx} = f \]
\[u(x, 0) = u_t(x, 0) = 0 = u_t - u_x = f u(x, 0) = r_1(1 - t) + u(\xi, \xi + t - 1) d\xi = 0 \cdot u_t + u_x = f u(x, 0) = u(0, t) = 0 \]

Green's Operator:

\[G = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \]
Unbounded wave equation:
Unbounded wave equation:

\[(D_{tt} - D_{xx}, [L_t, L_tD_t]) = (D_t - D_x, [L_t]) \cdot (D_t + D_x, [L_t])\]
Unbounded wave equation:

\[(D_{tt} - D_{xx}, [L_t, L_t D_t]) = (D_t - D_x, [L_t]) \cdot (D_t + D_x, [L_t])\]

or

\[
\begin{align*}
  u_{tt} - u_{xx} &= f \\
  u(x, 0) &= u_t(x, 0) = 0
\end{align*}
\]

\[
\begin{align*}
  u_t - u_x &= f \\
  u(x, 0) &= 0
\end{align*}
\]

\[
\begin{align*}
  u_t + u_x &= f \\
  u(x, 0) &= 0
\end{align*}
\]
Unbounded wave equation:

\[ (D_{tt} - D_{xx}, [L_t, L_tD_t]) = (D_t - D_x, [L_t]) \cdot (D_t + D_x, [L_t]) \]

or

\[
\begin{align*}
  u_{tt} - u_{xx} &= f \\
  u(x,0) &= u_t(x,0) = 0
\end{align*}
\]

Green’s Operator: \( G = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^* \)
Unbounded wave equation:
\[
(D_{tt} - D_{xx}, [L_t, L_tD_t]) = (D_t - D_x, [L_t]) \cdot (D_t + D_x, [L_t])
\]
or
\[
\begin{align*}
  u_{tt} - u_{xx} &= f \\
  u(x, 0) &= u_t(x, 0) = 0
\end{align*}
\]
Green’s Operator: \( G = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^* A_x \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^*
\]
\[
= \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}^* A_x \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ -1/2 & 1/2 \end{pmatrix}^* A_x \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}^*
\]
Factorization Examples for LPDEs

Unbounded wave equation:

\[(D_{tt} - D_{xx}, [L_t, L_tD_t]) = (D_t - D_x, [L_t]) \cdot (D_t + D_x, [L_t])\]

or

\[
\begin{align*}
  u_{tt} - u_{xx} &= f \\
  u(x, 0) &= u_t(x, 0) = 0
\end{align*}
\]

Green’s Operator: \[G = (\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}) \cdot A_x \cdot (\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix})^* \]

Bounded wave equation:

\[(D_{tt} - D_{xx}, [L_t, L_tD_t, L_x, R_x]) = (D_t - D_x, [L_t, S]) \cdot (D_t + D_x, [L_t, L_x])\]
Factorization Examples for LPDEs

Unbounded wave equation:

\[(D_{tt} - D_{xx}, [L_t, L_tD_t]) = (D_t - D_x, [L_t]) \cdot (D_t + D_x, [L_t])\]

or

\[
\begin{align*}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= f \\
\left. u(x,0) \right|_{t=0} &= u_t(x,0) = 0
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} &= f \\
\left. u(x,0) \right|_{t=0} &= 0
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} &= f \\
\left. u(x,0) \right|_{t=0} &= 0
\end{align*}
\]

Green’s Operator: \( G = \left( \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right) A_x \left( \begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array} \right)^* \left( \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right)^* \\
= \left( \begin{array}{cc} 1 & 0 \\ 1/2 & 1/2 \end{array} \right)^* A_x \left( \begin{array}{cc} 1 & 0 \\ -1 & 2 \end{array} \right)^* \cdot \left( \begin{array}{cc} 1 & 0 \\ -1/2 & 1/2 \end{array} \right)^* A_x \left( \begin{array}{cc} 1 & 0 \\ 1 & 2 \end{array} \right)^*
\)

Bounded wave equation:

\[(D_{tt} - D_{xx}, [L_t, L_tD_t, L_x, R_x]) = (D_t - D_x, [L_t, S]) \cdot (D_t + D_x, [L_t, L_x])\]

or

\[
\begin{align*}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= f \\
\left. u(x,0) \right|_{t=0} &= u_t(x,0) = u(0, t) = u(1, t) = 0 \\
\int_{(1-t)}^{1} u(\xi, \xi + t - 1) \, d\xi &= 0
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} &= f \\
\left. u(x,0) \right|_{t=0} &= \int_{(1-t)}^{1} u(\xi, \xi + t - 1) \, d\xi = 0 \\
\left. u(x,0) \right|_{t=0} &= u(0, t) = u(0, t) = 0
\end{align*}
\]

Markus Rosenkranz
Differential Algebra for Boundary Problems
Factorization Examples for LPDEs

Unbounded wave equation:

\[(D_{tt} - D_{xx}, [L_t, L_tD_t]) = (D_t - D_x, [L_t]) \cdot (D_t + D_x, [L_t])\]

or

\[
\begin{align*}
  u_{tt} - u_{xx} &= f \\
  u(x, 0) &= u_t(x, 0) = 0
\end{align*}
\]

Green’s Operator: \( G = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} A_x \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1/2 & 1/2 \end{pmatrix} \)


Bounded wave equation:

\[(D_{tt} - D_{xx}, [L_t, L_tD_t, L_x, R_x]) = (D_t - D_x, [L_t, S]) \cdot (D_t + D_x, [L_t, L_x])\]

or

\[
\begin{align*}
  u_{tt} - u_{xx} &= f \\
  u(x, 0) &= u_t(x, 0) = u(0, t) = u(1, t) = 0
\end{align*}
\]

\[
\begin{align*}
  u_t - u_x &= f \\
  u(x, 0) &= \int_{(1-t)}^1 u(\xi, \xi + t - 1) \, d\xi = 0
\end{align*}
\]

Green’s Operator \( G = \sum_{i=0}^{[t]-1} \frac{(-1)^i}{2} \int_{(t-i-1)^+}^{t-i} \int_{(1-t)}^1 |(-1)^i(x - \frac{1}{2}) + (t - \tau - i - \frac{1}{2})| \, d\xi \, d\tau \)
Geometric Interpretation

\[ u_{tt} - u_{xx} = f(u(t, x, 0) = u_t(x, 0) = u(0, t) = u(1, t) = 0) \]

\[ u_t - u_x = f(u(x, 0) = r_1(1 - t) + u(\xi, \xi + t - 1) d\xi = 0 \]

\[ u_t + u_x = f(u(x, 0) = u(0, t) = 0 \]

\[ \frac{t}{0} = \frac{x}{0} = \frac{x}{1} \]

\[ L(1 - t) + L(\eta) f(\frac{\eta}{2} + (1 - \frac{\eta}{2}) (\frac{\eta}{x + t - \eta} d\eta) \]

Markus Rosenkranz
Differential Algebra for Boundary Problems
Geometric Interpretation

\[ u_{tt} - u_{xx} = f \]
\[ u(x, 0) = u_t(x, 0) = u(0, t) = u(1, t) = 0 \]

\[ u_t - u_x = f \]
\[ u(x, 0) = \int_{(1-t)_+}^1 u(\xi, \xi + t - 1) d\xi = 0 \]
\[ u_t + u_x = f \]
\[ u(x, 0) = u(0, t) = 0 \]
Geometric Interpretation

\[
\begin{align*}
    u_{tt} - u_{xx} &= f \\
    u(x, 0) &= u_t(x, 0) = u(0, t) = u(1, t) = 0
\end{align*}
\]

\[
\begin{align*}
    u_t - u_x &= f \\
    u(x, 0) &= \int_{1-t}^{1} u(\xi, \xi + t - 1) d\xi = 0 \\
    u_t + u_x &= f \\
    u(x, 0) &= u(0, t) = 0
\end{align*}
\]
Geometric Interpretation

\[ u_{tt} - u_{xx} = f \]
\[ u(x, 0) = u_t(x, 0) = u(0, t) = u(1, t) = 0 \]

\[ u_t - u_x = f \]
\[ u(x, 0) = \int_{(1-t)+}^{1} u(\xi, \xi + t - 1) \, d\xi = 0 \]

\[ u_t + u_x = f \]
\[ u(x, 0) = u(0, t) = 0 \]

\[ G_1 f(x, t) = \int_{(x-t)+}^{x} f(\xi, \xi - x + t) \, d\xi \]
\[ G_2 f(x, t) = \int_{x}^{x+t} (-1)^{\lfloor \eta \rfloor} f\left( \frac{1}{2} + (-1)^{\lfloor \eta \rfloor} \left( \frac{\eta}{2} - \frac{1}{2} \right), x + t - \eta \right) \, d\eta \]
Outline

1. Abstract Boundary Problems
2. Ordinary Integro-Differential Operators
3. Partial Integro-Differential Operators
4. Conclusion
Summary and Future Work

What has been achieved:

- Algebraic theory for linear boundary problems
- Operator algebras for integration
- Algorithms for LODE case

What needs to be done:

- Green's operators for classes of LPDEs
- Discrete analogs
- Nonlinear boundary problems?
Summary and Future Work

What has been achieved:

- Algebraic theory for linear boundary problems
Summary and Future Work

What has been achieved:

- Algebraic theory for linear boundary problems
- Operator algebras for integration

What needs to be done:

- Green’s operators for classes of LPDEs
- Discrete analogs
- Nonlinear boundary problems?

THANK YOU
What has been achieved:
  - Algebraic theory for linear boundary problems
  - Operator algebras for integration
  - Algorithms for LODE case

What needs to be done:
  - Green’s operators for classes of LPDEs
  - Discrete analogs
  - Nonlinear boundary problems?
Summary and Future Work

What has been achieved:

- Algebraic theory for linear boundary problems
- Operator algebras for integration
- Algorithms for LODE case

What needs to be done:
Summary and Future Work

What has been achieved:
- Algebraic theory for linear boundary problems
- Operator algebras for integration
- Algorithms for LODE case

What needs to be done:
- Green’s operators for classes of LPDEs
Summary and Future Work

What has been achieved:
- Algebraic theory for linear boundary problems
- Operator algebras for integration
- Algorithms for LODE case

What needs to be done:
- Green’s operators for classes of LPDEs
- Discrete analogs
Summary and Future Work

What has been achieved:
- Algebraic theory for linear boundary problems
- Operator algebras for integration
- Algorithms for LODE case

What needs to be done:
- Green’s operators for classes of LPDEs
- Discrete analogs
- Nonlinear boundary problems?
Summary and Future Work

What has been achieved:
  ● Algebraic theory for linear boundary problems
  ● Operator algebras for integration
  ● Algorithms for LODE case

What needs to be done:
  ● Green’s operators for classes of LPDEs
  ● Discrete analogs
  ● Nonlinear boundary problems?

THANK YOU
G. Regensburger, M. Rosenkranz.


M. Rosenkranz.

M. Rosenkranz, G. Regensburger.


Table of Contents

1. Overview
2. Classical Beam Deflection
3. Analytic Method
4. Connecting Differential Algebra with Boundary Values
5. Abstract Boundary Problems
6. Regularity and Green’s Operators
7. Composition of Boundary Problems
8. Dual Boundary Problems
9. Determination of Green’s Operators
10. Factorization of Boundary Problems
11. Incarnations of Boundary Problems
12. Boundary Data and Boundary Values
13. Interpolator and Green’s Operators
14. LODE Example: Two-Point Boundary Problem
15. LPDE Example: Cauchy Problem
16. Integro-Differential Algebras
17. Alternative Characterizations
18. Univariate Operator Ring
19. Stieltjes Conditions versus Two-Point Conditions
20. Concrete Boundary Problems for LODEs
21. LODE Example Revisited
22. Third-Order Example
23. Factorization of Ordinary Boundary Problems
24. Basic Example: Smooth Functions
25. Action of Integrals and Substitutions
26. Notation for Special Matrices
27. Substitutive Algebras
28. Ordinary Rota-Baxter Algebras
29. Hierarchical Rota-Baxter Algebras
30. Verification of the Horizontal and Vertical Rule
31. Simple Properties
32. Admissible Coefficient Algebras
33. Coalgebra Structure for Coefficients
34. Normalization of General Line Integrators
35. Ordering of General Line Integrators
36. Coalescence of General Line Integrators
37. The Operator Ring
38. Natural Action and Termination
39. Conjectured Canonical Forms
40. Additional Rules for Derivations
41. LPDE Example Revisited
42. Factorization Examples for LPDEs
43. Geometric Interpretation
44. Summary and Future Work
45. References I
46. References II