Jacobi algebras, in-between Poisson, differential, and Lie algebras

Laurent Poinsot

LIPN, University Paris XIII, Sorbonne Paris Cité
France

Friday, May 15, 2015

---

1Second address: CReA, French Air Force Academy, France.
Table of contents

1 Motivations

2 A glance at universal (differential) algebra

3 Differential Lie algebras and their enveloping differential algebras

4 The embedding problem

5 Jacobi, Poisson and Lie-Rinehart algebras

6 Kirillov’s local Lie algebras and Lie algebroids
Let $R$ be a commutative ring with a unit.

A Lie algebra $(g, [-, -])$ is the data of a $R$-module $g$ and a bilinear map $[-, -]: g \times g \to g$, called the Lie bracket, such that

- It is alternating: $[x, x] = 0$ for every $x \in g$.

- It satisfies the Jacobi identity

  $$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

  for each $x, y, z \in g$.

A Lie algebra is said to be commutative whenever its bracket is the zero map.
Universal enveloping algebra

Any (say unital and associative) algebra \((A, \cdot)\) may be turned into a Lie algebra when equipped with the **commutator bracket**

\[
[x, y] = x \cdot y - y \cdot x .
\]
Universal enveloping algebra

Any (say unital and associative) algebra \((A, \cdot)\) may be turned into a Lie algebra when equipped with the **commutator bracket**

\[
[x, y] = x \cdot y - y \cdot x.
\]

Actually this defines a functor from the category \textbf{Ass} to the category \textbf{Lie}.

This functor admits a **left adjoint** namely the **universal enveloping algebra** \(U(g)\) of a Lie algebra \(g\).
Universal enveloping algebra

Any (say unital and associative) algebra \((A, \cdot)\) may be turned into a Lie algebra when equipped with the **commutator bracket**

\[
[x, y] = x \cdot y - y \cdot x.
\]

Actually this defines a functor from the category \(\text{Ass}\) to the category \(\text{Lie}\).

This functor admits a left adjoint namely the **universal enveloping algebra** \(\mathcal{U}(g)\) of a Lie algebra \(g\).

One has

\[
\mathcal{U}(g) \simeq T(g)/\langle xy -yx - [x, y]: x, y \in g \rangle
\]

where \(T(M)\) is the **tensor algebra** of a \(R\)-module \(M\).
Poincaré-Birkhoff-Witt theorem

Let $\mathfrak{g}$ be a Lie algebra (over $R$).

Let $j : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ be the Lie map defined as the composition $\mathfrak{g} \hookrightarrow T(\mathfrak{g}) \xrightarrow{\pi} \mathcal{U}(\mathfrak{g})$ (where $\pi$ is the canonical projection, and $\mathcal{U}(\mathfrak{g})$ is seen as a Lie algebra under its commutator bracket).

PBW Theorem

If $R$ is a field, then $j$ is one-to-one.

More generally, P.M. Cohn proved in 1963 that if the underlying abelian group of $\mathfrak{g}$ is torsion-free, then $j$ is one-to-one.

Remark

Actually, PBW theorem states that the associated graded algebra of $\mathcal{U}(\mathfrak{g})$ and the symmetric algebra of $\mathfrak{g}$ are isomorphic.
Poincaré-Birkhoff-Witt theorem

Let $\mathfrak{g}$ be a Lie algebra (over $R$).

Let $j: \mathfrak{g} \rightarrow \mathcal{U}(\mathfrak{g})$ be the Lie map defined as the composition $\mathfrak{g} \hookrightarrow T(\mathfrak{g}) \xrightarrow{\pi} \mathcal{U}(\mathfrak{g})$ (where $\pi$ is the canonical projection, and $\mathcal{U}(\mathfrak{g})$ is seen as a Lie algebra under its commutator bracket).

PBW Theorem

If $R$ is a field, then $j$ is one-to-one.

More generally, P.M. Cohn proved in 1963 that if the underlying abelian group of $\mathfrak{g}$ is torsion-free, then $j$ is one-to-one.
Poincaré-Birkhoff-Witt theorem

Let $\mathfrak{g}$ be a Lie algebra (over $R$).

Let $j: \mathfrak{g} \to U(\mathfrak{g})$ be the Lie map defined as the composition $\mathfrak{g} \hookrightarrow T(\mathfrak{g}) \xrightarrow{\pi} U(\mathfrak{g})$ (where $\pi$ is the canonical projection, and $U(\mathfrak{g})$ is seen as a Lie algebra under its commutator bracket).

**PBW Theorem**

If $R$ is a field, then $j$ is one-to-one.

More generally, P.M. Cohn proved in 1963 that if the underlying abelian group of $\mathfrak{g}$ is torsion-free, then $j$ is one-to-one.

**Remark**

Actually, PBW theorem states that the associated graded algebra of $U(\mathfrak{g})$ and the symmetric algebra of $\mathfrak{g}$ are isomorphic.
Is there a way to extend the notion of universal enveloping algebra to the differential setting?

Yes. And even (at least) two different ways. The first one is a somewhat "trivial" extension. Indeed, a derivation on an algebra is also a derivation for its commutator bracket. Moreover the universal enveloping algebra may be equipped with a (universal) derivation that extends the derivation of the Lie algebra, and the Poincaré-Birkhoff-Witt theorem remains unchanged.

The other one is rather different (since it is not based on the commutator) and is sketched hereafter.
Is there a way to extend the notion of universal enveloping algebra to the differential setting?

Yes. And even (at least) two different ways.
Is there a way to extend the notion of universal enveloping algebra to the differential setting?

Yes. And even (at least) two different ways.

The first one is a somewhat “trivial” extension. Indeed, a derivation on an algebra is also a derivation for its commutator bracket. Moreover the universal enveloping algebra may be equipped with a (universal) derivation that extends the derivation of the Lie algebra, and the Poincaré-Birkhoff-Witt theorem remains unchanged.
Is there a way to extend the notion of universal enveloping algebra to the differential setting?

Yes. And even (at least) two different ways.

The first one is a somewhat “trivial” extension. Indeed, a derivation on an algebra is also a derivation for its commutator bracket. Moreover the universal enveloping algebra may be equipped with a (universal) derivation that extends the derivation of the Lie algebra, and the Poincaré-Birkhoff-Witt theorem remains unchanged.

The other one is rather different (since it is not based on the commutator) and is sketched hereafter.
Wronskian bracket

Now, let us assume that \((A, \cdot, d)\) is a differential commutative algebra.

There is another Lie bracket given by the Wronskian

\[ W(x, y) = x \cdot d(y) - d(x) \cdot y \]

which turns \(A\) into a (differential) Lie algebra.

The above correspondence is actually functorial.
Wronskian bracket

Now, let us assume that \((A, \cdot, d)\) is a differential commutative algebra.

There is another Lie bracket given by the Wronskian

\[
W(x, y) = x \cdot d(y) - d(x) \cdot y
\]

which turns \(A\) into a (differential) Lie algebra.

The above correspondence is actually functorial. Whence one can ask a few questions:

1. Does it admit a left adjoint? In other terms, is there a universal enveloping differential (commutative) algebra? (Call it the Wronskian enveloping algebra.)
Wronskian bracket

Now, let us assume that \((A, \cdot, d)\) is a differential \textit{commutative} algebra.

There is another Lie bracket given by the \textit{Wronskian}

\[
W(x, y) = x \cdot d(y) - d(x) \cdot y
\]

which turns \(A\) into a (differential) Lie algebra.

The above correspondence is actually \textit{functorial}. Whence one can ask a few questions:

1. Does it admit a left adjoint? In other terms, is there a universal enveloping differential (commutative) algebra? (Call it the \textit{Wronskian enveloping algebra}.) Yes.
Wronskian bracket

Now, let us assume that $(A, \cdot, d)$ is a differential commutative algebra.

There is another Lie bracket given by the Wronskian

$$W(x, y) = x \cdot d(y) - d(x) \cdot y$$

which turns $A$ into a (differential) Lie algebra.

The above correspondence is actually functorial. Whence one can ask a few questions:

1. Does it admit a left adjoint? In other terms, is there a universal enveloping differential (commutative) algebra? (Call it the Wronskian enveloping algebra.) Yes.
2. Under which assumptions the canonical map from a Lie algebra to its differential enveloping algebra is one-to-one?
Wronskian bracket

Now, let us assume that \((A, \cdot, d)\) is a differential commutative algebra.

There is another Lie bracket given by the Wronskian

\[ W(x, y) = x \cdot d(y) - d(x) \cdot y \]

which turns \(A\) into a (differential) Lie algebra.

The above correspondence is actually functorial. Whence one can ask a few questions:

1. Does it admit a left adjoint? In other terms, is there a universal enveloping differential (commutative) algebra? (Call it the Wronskian enveloping algebra.) Yes.

2. Under which assumptions the canonical map from a Lie algebra to its differential enveloping algebra is one-to-one? Unfortunatly, I don’t know a general answer yet.
Wronskian bracket

Now, let us assume that \((A, \cdot, d)\) is a differential commutative algebra.

There is another Lie bracket given by the Wronskian

\[
W(x, y) = x \cdot d(y) - d(x) \cdot y
\]

which turns \(A\) into a (differential) Lie algebra.

The above correspondence is actually functorial. Whence one can ask a few questions:

1. Does it admit a left adjoint? In other terms, is there a universal enveloping differential (commutative) algebra? (Call it the Wronskian enveloping algebra.) Yes.

2. Under which assumptions the canonical map from a Lie algebra to its differential enveloping algebra is one-to-one? Unfortunately, I don’t know a general answer yet.

In this talk I will also provide some examples of embedding / non-embedding of Lie algebras into their differential associative envelope.
Table of contents

1 Motivations

2 A glance at universal (differential) algebra

3 Differential Lie algebras and their enveloping differential algebras

4 The embedding problem

5 Jacobi, Poisson and Lie-Rinehart algebras

6 Kirillov’s local Lie algebras and Lie algebroids
Universal algebra

An operator domain or signature is a $\mathbb{N}$-graded set $\Sigma$, i.e., a family of sets $(\Sigma(n))_{n \in \mathbb{N}}$. 

Examples

Monoids are $\Sigma$-algebras for the signature $\Sigma(0) = \{e\}$, $\Sigma(2) = \{\ast\}$, $\Sigma(n) = \emptyset$, $n \neq 0, 2$.

Groups are $\Sigma$-algebras for the signature $\Sigma(0) = \{e\}$, $\Sigma(1) = \{(-1)^{-1}\}$, $\Sigma(2) = \{\ast\}$, $\Sigma(n) = \emptyset$, $n \neq 0, 1, 2$.

There are signatures for (associative) $\mathcal{R}$-algebras, Lie $\mathcal{R}$-algebras, and their differential counterparts.
Universal algebra

An operator domain or signature is a $\mathbb{N}$-graded set $\Sigma$, i.e., a family of sets $(\Sigma(n))_{n \in \mathbb{N}}$. The members of $\Sigma(0)$ are referred to as symbols of constant, while those of $\Sigma(n), n > 0$, are called symbols of (n-ary) functions.
Universal algebra

An operator domain or signature is a \( \mathbb{N} \)-graded set \( \Sigma \), i.e., a family of sets \((\Sigma(n))_{n \in \mathbb{N}}\). The members of \( \Sigma(0) \) are referred to as symbols of constant, while those of \( \Sigma(n), n > 0 \), are called symbols of (n-ary) functions.

A \( \Sigma \)-algebra is a pair \((A, F)\), where \( A \) is a set, and \( F \) is a family of set-theoretic maps \((F(n): \Sigma(n) \to A^{A^n})_n\) that makes possible to interpret the symbols of functions (resp., constants) by \( n \)-ary functions on (resp., members of) \( A \).
Universal algebra

An operator domain or signature is a \( \mathbb{N} \)-graded set \( \Sigma \), i.e., a family of sets \( (\Sigma(n))_{n \in \mathbb{N}} \). The members of \( \Sigma(0) \) are referred to as symbols of constant, while those of \( \Sigma(n), n > 0 \), are called symbols of (n-ary) functions.

A \( \Sigma \)-algebra is a pair \( (A, F) \), where \( A \) is a set, and \( F \) is a family of set-theoretic maps \( (F(n) : \Sigma(n) \rightarrow A^{A^n})_n \) that makes possible to interpret the symbols of functions (resp., constants) by \( n \)-ary functions on (resp., members of) \( A \).

Examples

- Monoids are \( \Sigma \)-algebras for the signature \( \Sigma(0) = \{ e \} \), \( \Sigma(2) = \{ * \} \), \( \Sigma(n) = \emptyset, n \neq 0, 2 \).
Universal algebra

An **operator domain** or **signature** is a $\mathbb{N}$-graded set $\Sigma$, i.e., a family of sets $(\Sigma(n))_{n \in \mathbb{N}}$. The members of $\Sigma(0)$ are referred to as **symbols of constant**, while those of $\Sigma(n)$, $n > 0$, are called **symbols of (n-ary) functions**.

A $\Sigma$-**algebra** is a pair $(A, F)$, where $A$ is a set, and $F$ is a family of set-theoretic maps $(F(n) : \Sigma(n) \rightarrow A^{A^n})_n$ that makes possible to interpret the symbols of functions (resp., constants) by $n$-ary functions on (resp., members of) $A$.

**Examples**

- **Monoids** are $\Sigma$-algebras for the signature $\Sigma(0) = \{ e \}$, $\Sigma(2) = \{ \ast \}$, $\Sigma(n) = \emptyset$, $n \neq 0, 2$.

- **Groups** are $\Sigma$-algebras for the signature $\Sigma(0) = \{ e \}$, $\Sigma(1) = \{ (-)^{-1} \}$, $\Sigma(2) = \{ \ast \}$, $\Sigma(n) = \emptyset$, $n \neq 0, 1, 2$. 
Universal algebra

An operator domain or signature is a $\mathbb{N}$-graded set $\Sigma$, i.e., a family of sets $(\Sigma(n))_{n \in \mathbb{N}}$. The members of $\Sigma(0)$ are referred to as symbols of constant, while those of $\Sigma(n)$, $n > 0$, are called symbols of (n-ary) functions.

A $\Sigma$-algebra is a pair $(A, F)$, where $A$ is a set, and $F$ is a family of set-theoretic maps $(F(n): \Sigma(n) \to A^A^n)_n$ that makes possible to interpret the symbols of functions (resp., constants) by $n$-ary functions on (resp., members of) $A$.

Examples

- Monoids are $\Sigma$-algebras for the signature $\Sigma(0) = \{ e \}$, $\Sigma(2) = \{ * \}$, $\Sigma(n) = \emptyset$, $n \neq 0, 2$.
- Groups are $\Sigma$-algebras for the signature $\Sigma(0) = \{ e \}$, $\Sigma(1) = \{ (-)^{-1} \}$, $\Sigma(2) = \{ * \}$, $\Sigma(n) = \emptyset$, $n \neq 0, 1, 2$.
- There are signatures for (associative) $R$-algebras, Lie $R$-algebras, and their differential counterparts.
Equational varieties

A class $\mathcal{V}$ of $\Sigma$-algebras is said to be an equational variety when each member of the class satisfies some given axioms or identities.
Equational varieties

A class $\mathcal{V}$ of $\Sigma$-algebras is said to be an **equational variety** when each member of the class satisfies some given **axioms** or **identities**.

**Example**

Equations for monoids: $x \ast e = e = e \ast x$, $(x \ast y) \ast z = x \ast (y \ast z)$. 
Equational varieties

A class $\mathbf{V}$ of $\Sigma$-algebras is said to be an equational variety when each member of the class satisfies some given axioms or identities.

Example

Equations for monoids: $x \ast e = e = e \ast x$, $(x \ast y) \ast z = x \ast (y \ast z)$.

Each variety of $\Sigma$-algebras with its homomorphisms (maps preserving the structural operations) forms a category.
Equational varieties

A class $V$ of $\Sigma$-algebras is said to be an equational variety when each member of the class satisfies some given axioms or identities.

Example

Equations for monoids: $x * e = e = e * x$, $(x * y) * z = x * (y * z)$.

Each variety of $\Sigma$-algebras with its homomorphisms (maps preserving the structural operations) forms a category.

Some (counter-)examples

- Semigroups, inverse semigroups, monoids, commutative monoids, groups, abelian groups, rings, $R$-algebras for a unital commutative ring $R$, Lie $R$-algebras, Jordan $R$-algebras, etc.
Equational varieties

A class $\mathbf{V}$ of $\Sigma$-algebras is said to be an equational variety when each member of the class satisfies some given axioms or identities.

Example

Equations for monoids: $x \ast e = e = e \ast x$, $(x \ast y) \ast z = x \ast (y \ast z)$.

Each variety of $\Sigma$-algebras with its homomorphisms (maps preserving the structural operations) forms a category.

Some (counter-)examples

- Semigroups, inverse semigroups, monoids, commutative monoids, groups, abelian groups, rings, $R$-algebras for a unital commutative ring $R$, Lie $R$-algebras, Jordan $R$-algebras, etc.
- Fields (inversion is only partially defined), small categories, and the category of monoids with invertible elements (groups!), because it is not closed under sub-algebras (e.g., the sub-monoid $\mathbb{N}$ of $\mathbb{Z}$).
Algebraic functors

One of the key features of equational varieties is the fact that they come equipped with a forgetful functor $U_V : V \to \text{Set}$ (it maps an algebra to its carrier set). So they are concrete categories over $\text{Set}$ (and even monadic).

Theorem (Bill Lawvere)

Any algebraic functor admits a left adjoint. In particular the forgetful functor $U_V$ itself has a left adjoint. Hence for any set $X$, there exists a free algebra $V(X)$ in the variety $V$. By this is meant that there is a universal map $\eta_X : X \to V(X)$ such that for each algebra $(A, F)$ in the variety $V$, and for each set-theoretic map $f : X \to A$, there exists a unique homomorphism of algebras $\hat{f} : V(X) \to (A, F)$ such that $\hat{f} \circ \eta_X = f$.

It is also well-known that $\eta_X$ is one-to-one whenever $V$ is a non-trivial variety (i.e., there are algebras with more than one element in the variety $V$).
Algebraic functors

One of the key features of equational varieties is the fact that they come equipped with a forgetful functor $U_V : V \to \text{Set}$ (it maps an algebra to its carrier set). So they are concrete categories over $\text{Set}$ (and even monadic).

Let $V$ and $W$ be two equational varieties of algebras (not necessarily over the same signature). A functor $F : V \to W$ is said to be an algebraic functor if it preserves the forgetful functors, i.e., $U_W \circ F = U_V$.

Theorem (Bill Lawvere)
Any algebraic functor admits a left adjoint. In particular the forgetful functor $U_V$ itself has a left adjoint. Hence for any set $X$, there exists a free algebra $V(X)$ in the variety $V$. By this is meant that there is a universal map $\eta_X : X \to V(X)$ such that for each algebra $(A, F)$ in the variety $V$, and for each set-theoretic map $f : X \to A$, there exists a unique homomorphism of algebras $\hat{f} : V(X) \to (A, F)$ such that $\hat{f} \circ \eta_X = f$.

It is also well-known that $\eta_X$ is one-to-one whenever $V$ is a non-trivial variety (i.e., there are algebras with more than one element in the variety $V$).
Algebraic functors

One of the key features of equational varieties is the fact that they come equipped with a forgetful functor $U_V : V \to \text{Set}$ (it maps an algebra to its carrier set). So they are concrete categories over $\text{Set}$ (and even monadic).

Let $V$ and $W$ be two equational varieties of algebras (not necessarily over the same signature). A functor $F : V \to W$ is said to be an algebraic functor if it preserves the forgetful functors, i.e., $U_W \circ F = U_V$.

**Theorem (Bill Lawvere)**

Any algebraic functor admits a left adjoint.
Algebraic functors

One of the key features of equational varieties is the fact that they come equipped with a forgetful functor $U_V : V \rightarrow \text{Set}$ (it maps an algebra to its carrier set). So they are concrete categories over $\text{Set}$ (and even monadic).

Let $V$ and $W$ be two equational varieties of algebras (not necessarily over the same signature). A functor $F : V \rightarrow W$ is said to be an algebraic functor if it preserves the forgetful functors, i.e., $U_W \circ F = U_V$.

**Theorem (Bill Lawvere)**

Any algebraic functor admits a left adjoint.

In particular the forgetful functor $U_V$ itself has a left adjoint. Hence for any set $X$, there exists a free algebra $V(X)$ in the variety $V$. 
Algebraic functors

One of the key features of equational varieties is the fact that they come equipped with a forgetful functor $U_V : V \to \text{Set}$ (it maps an algebra to its carrier set). So they are concrete categories over $\text{Set}$ (and even monadic).

Let $V$ and $W$ be two equational varieties of algebras (not necessarily over the same signature). A functor $F : V \to W$ is said to be an algebraic functor if it preserves the forgetful functors, i.e., $U_W \circ F = U_V$.

**Theorem (Bill Lawvere)**

Any algebraic functor admits a left adjoint.

In particular the forgetful functor $U_V$ itself has a left adjoint. Hence for any set $X$, there exists a free algebra $V(X)$ in the variety $V$. By this is meant that there is a universal map $\eta_X : X \to V(X)$ such that for each algebra $(A, F)$ in the variety $V$, and for each set-theoretic map $f : X \to A$, there exists a unique homomorphism of algebras $\hat{f} : V(X) \to (A, F)$ such that $\hat{f} \circ \eta_X = f$. 
Algebraic functors

One of the key features of equational varieties is the fact that they come equipped with a forgetful functor $U_V : V \rightarrow \text{Set}$ (it maps an algebra to its carrier set). So they are concrete categories over $\text{Set}$ (and even monadic).

Let $V$ and $W$ be two equational varieties of algebras (not necessarily over the same signature). A functor $F : V \rightarrow W$ is said to be an algebraic functor if it preserves the forgetful functors, i.e., $U_W \circ F = U_V$.

**Theorem (Bill Lawvere)**

Any algebraic functor admits a left adjoint.

In particular the forgetful functor $U_V$ itself has a left adjoint. Hence for any set $X$, there exists a free algebra $V(X)$ in the variety $V$. By this is meant that there is a universal map $\eta_X : X \rightarrow V(X)$ such that for each algebra $(A, F)$ in the variety $V$, and for each set-theoretic map $f : X \rightarrow A$, there exists a unique homomorphism of algebras $\hat{f} : V(X) \rightarrow (A, F)$ such that $\hat{f} \circ \eta_X = f$. It is also well-known that $\eta_X$ is one-to-one whenever $V$ is a non-trivial variety (i.e., there are algebras with more than one element in the variety $V$).
Generalities about differential algebras

Let $R$ be a commutative ring with a unit.

Let $V$ be a variety of (not necessarily associative nor unital) $R$-algebras (i.e., $R$-modules $M$ with a $R$-bilinear operation $\cdot : M \times M \to M$ subject to some additional axioms).

For $V$ one may have in mind $\text{Ass}$ or $\text{Lie}$. 
Generalities about differential algebras

Let $R$ be a commutative ring with a unit.

Let $V$ be a variety of (not necessarily associative nor unital) $R$-algebras (i.e., $R$-modules $M$ with a $R$-bilinear operation $\cdot : M \times M \to M$ subject to some additional axioms).

For $V$ one may have in mind $\text{Ass}$ or $\text{Lie}$.

A derivation $d : M \to M$ is a $R$-linear map that satisfies Leibniz identity

$$d(x \cdot y) = d(x) \cdot y + x \cdot d(y).$$
Generalities about differential algebras

Let $R$ be a commutative ring with a unit.

Let $V$ be a variety of (not necessarily associative nor unital) $R$-algebras (i.e., $R$-modules $M$ with a $R$-bilinear operation $\cdot : M \times M \to M$ subject to some additional axioms).

For $V$ one may have in mind $\text{Ass}$ or $\text{Lie}$.

A derivation $d : M \to M$ is a $R$-linear map that satisfies Leibniz identity

$$d(x \cdot y) = d(x) \cdot y + x \cdot d(y).$$

By considering algebras $(M, \cdot)$ of $V$ with a derivation $d$ and homomorphisms of algebras commuting with derivations, one gets a variety, say $\text{Diff}V$, of differential algebras (in $V$).
A two-sided (differential) ideal \( I \) of a differential algebra \((M, \cdot, d)\) is just a two-sided ideal of \((M, \cdot)\) (i.e., a sub-module such that \( M \cdot I \subseteq I \supseteq I \cdot M \)) such that \( d(I) \subseteq I \).
A two-sided (differential) ideal $I$ of a differential algebra $(M, \cdot, d)$ is just a two-sided ideal of $(M, \cdot)$ (i.e., a sub-module such that $M \cdot I \subseteq I \supseteq I \cdot M$) such that $d(I) \subseteq I$.

It turns out that $M/I$ becomes a differential algebra with derivation $\tilde{d}(x + I) = d(x) + I$ and the natural epimorphism $M \to M/I$ is a homomorphism of differential algebras.
Differential ideals

A two-sided (differential) ideal $I$ of a differential algebra $(M, \cdot, d)$ is just a two-sided ideal of $(M, \cdot)$ (i.e., a sub-module such that $M \cdot I \subseteq I \supseteq I \cdot M$) such that $d(I) \subseteq I$.

It turns out that $M/I$ becomes a differential algebra with derivation $\tilde{d}(x + I) = d(x) + I$ and the natural epimorphism $M \to M/I$ is a homomorphism of differential algebras.

Because an intersection of any family of differential ideals also is a differential ideal, it makes also sense to talk about the least differential ideal generated by a set.
A forgetful functor (1/2)
The free differential algebra generated by an algebra

There is an obvious forgetful functor $\text{Diff}_V \to V$ which admits a left adjoint (since it is an algebraic functor).

Hence any algebra in $V$ “freely generates” a differential algebra (in $V$).
There is an obvious forgetful functor $\text{Diff}V \to V$ which admits a left adjoint (since it is an algebraic functor).

Hence any algebra in $V$ “freely generates” a differential algebra (in $V$).

**The construction:** let $(M, \cdot)$ be an algebra in $V$. Let $\text{FDiff}V(|M|)$ be the free differential algebra in $V$ generated by the set $|M|$ (carrier set of $(M, \cdot)$), and let $j: |M| \to |\text{FDiff}V(|M|)|$ be the canonical map.
A forgetful functor (1/2)
The free differential algebra generated by an algebra

There is an obvious forgetful functor $\text{Diff}_V \rightarrow V$ which admits a left adjoint (since it is an algebraic functor).

Hence any algebra in $V$ “freely generates” a differential algebra (in $V$).

The construction: let $(M, \cdot)$ be an algebra in $V$. Let $\text{FDiff}_V(|M|)$ be the free differential algebra in $V$ generated by the set $|M|$ (carrier set of $(M, \cdot)$), and let $j: |M| \rightarrow |\text{FDiff}_V(|M|)|$ be the canonical map. Let $I$ be the differential ideal generated by $j(x + y) - j(x) - j(y)$, $j(x \cdot y) - j(x)j(y)$, $j(rx) - rj(x)$, $x, y \in |M|$, $r \in R$. 
A forgetful functor (1/2)
The free differential algebra generated by an algebra

There is an obvious forgetful functor $\text{DiffV} \to \mathbf{V}$ which admits a left adjoint (since it is an algebraic functor).

Hence any algebra in $\mathbf{V}$ “freely generates” a differential algebra (in $\mathbf{V}$).

The construction: let $(M, \cdot)$ be an algebra in $\mathbf{V}$. Let $F\text{DiffV}(|M|)$ be the free differential algebra in $\mathbf{V}$ generated by the set $|M|$ (carrier set of $(M, \cdot)$), and let $j: |M| \to |F\text{DiffV}(|M|)|$ be the canonical map. Let $I$ be the differential ideal generated by $j(x + y) - j(x) - j(y)$, $j(x \cdot y) - j(x)j(y)$, $j(rx) - rj(x)$, $x, y \in |M|$, $r \in R$.

Then, $F\text{DiffV}(|M|)/I$ is the free differential algebra generated by $(M, \cdot)$. 
Let \((N, \cdot, e)\) be a differential algebra in \(V\), and let \(\phi: (M, \cdot) \to (N, \cdot)\) be an algebra map.
Let \((N, \cdot, e)\) be a differential algebra in \(\mathbf{V}\), and let \(\phi: (M, \cdot) \to (N, \cdot)\) be an algebra map.

Let \(\hat{\phi}: \text{FDiff}_V(|M|) \to (N, \cdot, e)\) be the unique differential algebra map such that \(\hat{\phi} \circ j = \phi\).
Let $(N, \cdot, e)$ be a differential algebra in $V$, and let $\phi: (M, \cdot) \rightarrow (N, \cdot)$ be an algebra map.

Let $\hat{\phi}: \text{FDiff}_V(|M|) \rightarrow (N, \cdot, e)$ be the unique differential algebra map such that $\hat{\phi} \circ j = \phi$.

Of course $I \subseteq \ker \hat{\phi}$ (since $\phi$ is an algebra map).
Let \((N, \cdot, e)\) be a differential algebra in \(V\), and let \(\phi: (M, \cdot) \rightarrow (N, \cdot)\) be an algebra map.

Let \(\hat{\phi}: FDiffV(|M|) \rightarrow (N, \cdot, e)\) be the unique differential algebra map such that \(\hat{\phi} \circ j = \phi\).

Of course \(I \subseteq \ker \hat{\phi}\) (since \(\phi\) is an algebra map).

Hence there is a unique differential algebra map \(\tilde{\phi}: FDiffV(|M|)/I \rightarrow (N, \cdot, e)\) such that \(\tilde{\phi} \circ \pi \circ j = \phi\).
Example

The free differential Lie algebra generated by a Lie algebra \( V = \text{Lie} \) in order to obtain the free differential Lie algebra \( \mathcal{DL}(\mathfrak{g}) := FDiffLie(|\mathfrak{g}|)/I \) generated by a Lie algebra \( \mathfrak{g} \).
Example

The free differential Lie algebra generated by a Lie algebra \( V = \text{Lie} \) in order to obtain the free differential Lie algebra \( DL(g) := FDiffLie(|g|)/I \) generated by a Lie algebra \( g \).

It is easily seen that any algebra \( g \) canonically embeds into its differential envelope \( DL(g) \) (because \( (g, 0) \) is itself a differential Lie algebra).
Example

The free differential Lie algebra generated by a Lie algebra \( V = \text{Lie} \) in order to obtain the free differential Lie algebra \( DL(g) := FDiffLie(|g|)/I \) generated by a Lie algebra \( g \).

It is easily seen that any algebra \( g \) canonically embeds into its differential envelope \( DL(g) \) (because \((g, 0)\) is itself a differential Lie algebra).

One can even describe \( FDiffLie(X) \) for a set \( X \):
Example

The free differential Lie algebra generated by a Lie algebra \( \mathfrak{g} \) / by a set

One may apply the results from the previous slide with \( \mathbf{V} = \mathbf{Lie} \) in order to obtain the free differential Lie algebra \( \mathcal{DL}(\mathfrak{g}) := FDiffLie(|\mathfrak{g}|)/I \) generated by a Lie algebra \( \mathfrak{g} \).

It is easily seen that any algebra \( \mathfrak{g} \) canonically embeds into its differential envelope \( \mathcal{DL}(\mathfrak{g}) \) (because \( (\mathfrak{g},0) \) is itself a differential Lie algebra).

One can even describe \( FDiffLie(X) \) for a set \( X \): Let \( M_X \) be the free magma on the set \( X \times \mathbb{N} \), and let \( A_X := RM_X \) be the free \( R \)-module generated by \( M_X \).
Example

The free differential Lie algebra generated by a Lie algebra / by a set

One may apply the results from the previous slide with $V = \text{Lie}$ in order to obtain the free differential Lie algebra $\mathcal{DL}(\mathfrak{g}) := \text{FDiffLie}(\mathfrak{g})/I$ generated by a Lie algebra $\mathfrak{g}$.

It is easily seen that any algebra $\mathfrak{g}$ canonically embeds into its differential envelope $\mathcal{DL}(\mathfrak{g})$ (because $(\mathfrak{g}, 0)$ is itself a differential Lie algebra).

One can even describe $\text{FDiffLie}(X)$ for a set $X$: Let $M_X$ be the free magma on the set $X \times \mathbb{N}$, and let $A_X := RM_X$ be the free $R$-module generated by $M_X$.

$A_X$ is a (non associative) algebra with bilinear multiplication extending the product in $M_X$. It is even the free (non associative) differential algebra with derivation $d$ given on generators $(x, i)$ by $d(x, i) := (x, i + 1)$, $x \in X$, $i \in \mathbb{N}$.
Example

The free differential Lie algebra generated by a Lie algebra $\mathfrak{g}$ / by a set $X$.

One may apply the results from the previous slide with $V = \text{Lie}$ in order to obtain the free differential Lie algebra $\mathcal{DL}(\mathfrak{g}) := \text{FDiffLie}(\mathfrak{g})/I$ generated by a Lie algebra $\mathfrak{g}$.

It is easily seen that any algebra $\mathfrak{g}$ canonically embeds into its differential envelope $\mathcal{DL}(\mathfrak{g})$ (because $(\mathfrak{g}, 0)$ is itself a differential Lie algebra).

One can even describe $\text{FDiffLie}(X)$ for a set $X$: Let $M_X$ be the free magma on the set $X \times \mathbb{N}$, and let $A_X := R M_X$ be the free $R$-module generated by $M_X$.

$A_X$ is a (non associative) algebra with bilinear multiplication extending the product in $M_X$. It is even the free (non associative) differential algebra with derivation $d$ given on generators $(x, i)$ by $d(x, i) := (x, i + 1)$, $x \in X$, $i \in \mathbb{N}$.

$\text{FDiffLie}(X) = A_X/J$, with the quotient derivation, where $J$ is the two-sided differential ideal of $A_X$ generated by $tt$, $(rs)t + (st)r + (tr)s$, $r, s, t \in M_X$. 

Another example
The free commutative differential algebra generated by an algebra / a set

The usual algebra $R\{X\}$ of differential polynomials in the (mutually commuting) variables $x \in X$ is the free commutative differential algebra generated by the set $X$. 
Another example
The free commutative differential algebra generated by an algebra / a set

The usual algebra $\mathcal{R}\{X\}$ of differential polynomials in the (mutually commuting) variables $x \in X$ is the free commutative differential algebra generated by the set $X$.

Let $A$ be a commutative (associative) algebra with a unit.
Another example

The free commutative differential algebra generated by an algebra / a set

The usual algebra $R\{X\}$ of differential polynomials in the (mutually commuting) variables $x \in X$ is the free commutative differential algebra generated by the set $X$.

Let $A$ be a commutative (associative) algebra with a unit. Then, $R\{|A|\}/I$, where $I$ is the two-sided differential ideal generated by the relations that would turn the canonical map $j: |A| \to R\{|A|\}$ into an algebra map, is the free commutative differential algebra generated by $A$. 

Remarks

A embeds, as sub-algebra, into $R\{|A|\}/I$ by $j$.

The above construction may be adapted for not necessarily commutative algebras.
Another example
The free commutative differential algebra generated by an algebra / a set

The usual algebra $R\{X\}$ of differential polynomials in the (mutually commuting) variables $x \in X$ is the free commutative differential algebra generated by the set $X$.

Let $A$ be a commutative (associative) algebra with a unit. Then, $R\{|A|\}/I$, where $I$ is the two-sided differential ideal generated by the relations that would turn the canonical map $j: |A| \to R\{|A|\}$ into an algebra map, is the free commutative differential algebra generated by $A$.

Remarks
- $A$ embeds, as sub-algebra, into $R\{|A|\}/I$ by $j$. 
Another example
The free commutative differential algebra generated by an algebra / a set

The usual algebra $R\{X\}$ of differential polynomials in the (mutually commuting) variables $x \in X$ is the free commutative differential algebra generated by the set $X$.

Let $A$ be a commutative (associative) algebra with a unit. Then, $R\{|A|\}/I$, where $I$ is the two-sided differential ideal generated by the relations that would turn the canonical map $j$: $|A| \to R\{|A|\}$ into an algebra map, is the free commutative differential algebra generated by $A$.

Remarks

- $A$ embeds, as sub-algebra, into $R\{|A|\}/I$ by $j$.
- The above construction may be adapted for not necessarily commutative algebras.
The variety $\mathbf{V}$ embeds into the variety $\text{DiffV}$ since any algebra in $\mathbf{V}$ may be seen as a differential algebra with the zero (or trivial) derivation.
The variety $\mathcal{V}$ embeds into the variety $\text{DiffV}$ since any algebra in $\mathcal{V}$ may be seen as a differential algebra with the zero (or trivial) derivation.

Of course this embedding preserves the forgetful functors, hence admits a left adjoint, i.e., $\mathcal{V}$ is a **reflective sub-category** of $\text{DiffV}$, this means that any differential algebra (in $\mathcal{V}$) “freely generates” an algebra in $\mathcal{V}$. 
Reflective sub-category (1/2)

\[ \mathbf{V} \hookrightarrow \mathbf{DiffV} \]

The variety \( \mathbf{V} \) embeds into the variety \( \mathbf{DiffV} \) since any algebra in \( \mathbf{V} \) may be seen as a differential algebra with the zero (or trivial) derivation.

Of course this embedding preserves the forgetful functors, hence admits a left adjoint, i.e., \( \mathbf{V} \) is a reflective sub-category of \( \mathbf{DiffV} \), this means that any differential algebra (in \( \mathbf{V} \)) “freely generates” an algebra in \( \mathbf{V} \).

The construction: let \( (M, \cdot, d) \) be a member of \( \mathbf{DiffV} \). Let \( I_d \) be the (algebraic) ideal generated \( im(d) \). Thus, \( M/I_d \) is a member of \( \mathbf{V} \), and the natural projection \( \pi: M \rightarrow M/I_d \) is a homomorphism of algebras.
Reflective sub-category (2/2)

Universal property

Given an algebra \((N, \cdot)\) and a homomorphism of differential algebras 
\(\phi: (M, \cdot, d) \rightarrow (N, \cdot, 0)\),
Given an algebra \((N, \cdot)\) and a homomorphism of differential algebras \(\phi: (M, \cdot, d) \rightarrow (N, \cdot, 0)\), because \(\phi \circ d = 0\), it passes to the quotient and gives rise to a unique homomorphism of algebras \(\hat{\phi}: (M/Id, \cdot) \rightarrow (N, \cdot)\) such that \(\hat{\phi} \circ \pi = \phi\).
Table of contents

1. Motivations

2. A glance at universal (differential) algebra

3. Differential Lie algebras and their enveloping differential algebras

4. The embedding problem

5. Jacobi, Poisson and Lie-Rinehart algebras

6. Kirillov’s local Lie algebras and Lie algebroids
Extension of the usual universal enveloping algebra to the differential setting

Let \((A, d)\) be a differential (associative) algebra.

\[
\begin{align*}
\text{DiffAss} & \quad \text{Comm} \\
& \quad \text{bracket} \\
\downarrow & \quad \downarrow \\
\text{DiffLie} & \quad \text{Ass} \\
& \quad \text{bracket} \\
& \quad \text{Lie}
\end{align*}
\]
Let \((A, d)\) be a differential (associative) algebra.

One has \(d([x, y]) = d(xy - yx) = d(x)y + xd(y) - d(y)x - yd(x) = [d(x), y] + [x, d(y)]\). Hence, \((A, [-, -], d)\) is a differential Lie algebra.
Extension of the usual universal enveloping algebra to the differential setting

Let \((A, d)\) be a differential (associative) algebra.

One has
\[
d([x, y]) = d(xy - yx) = d(xy) - d(y)x - xd(y) = [d(x), y] + [x, d(y)].
\]
Hence, \((A, [-, -], d)\) is a differential Lie algebra.

This gives rise to a functor \(\text{DiffAss} \to \text{DiffLie}\) which makes commute the following diagram (of forgetful functors).

\[
\begin{CD}
\text{DiffAss} @>\text{Comm. bracket}>> \text{DiffLie} \\
\downarrow \text{forgets der.} @. \downarrow \text{forgets der.} \\
\text{Ass} @>\text{Comm. bracket}>> \text{Lie}
\end{CD}
\]

All functors in this diagram admit a left adjoint.
A construction

Let \((g, [\mathbf{\cdot}, \mathbf{\cdot}], d)\) be a differential Lie algebra.

Let \(\partial\) be the unique derivation on \(T(g)\) that extends \(d\).
A construction

Let \((g, [-,-], d)\) be a differential Lie algebra.

Let \(\partial\) be the unique derivation on \(T(g)\) that extends \(d\). It satisfies
\[
\partial(xy - yx - [x,y]) = d(x)y + xd(y) - d(y)x - yd(x) - [d(x), y] - [x, d(y)]
\]
Let \((g, [-, -], d)\) be a differential Lie algebra.

Let \(\partial\) be the unique derivation on \(T(g)\) that extends \(d\). It satisfies
\[
\partial(xy - yx - [x, y]) = d(x)y + xd(y) - d(y)x - yd(x) - [d(x), y] - [x, d(y)] =
\]
\[
d(x)y - yd(x) - [d(x), y] + xd(y) - d(y)x - [x, d(y)],
\]
Let \((g, [-, -], d)\) be a differential Lie algebra.

Let \(\partial\) be the unique derivation on \(T(g)\) that extends \(d\). It satisfies

\[
\partial(xy - yx - [x, y]) = d(x)y + xd(y) - d(y)x - yd(x) - [d(x), y] - [x, d(y)] = d(x)y - yd(x) - [d(x), y] + xd(y) - d(y)x - [x, d(y)],
\]

so it factors as a linear map \(\tilde{\partial} : \mathcal{U}(g) \rightarrow \mathcal{U}(g)\) which is easily seen to be a derivation.
Universal property

\((U(g), \tilde{\partial})\) satisfies the following universal property:
\((\mathcal{U}(\mathfrak{g}), \tilde{\partial})\) satisfies the following universal property:

Let \((A, D)\) be a differential algebra, and let \(\phi: (\mathfrak{g}, [-, -], d) \to (A, [-, -], D)\) be a homomorphism of differential Lie algebras.
(\mathcal{U}(\mathfrak{g}), \tilde{\partial}) satisfies the following universal property:

Let \((\mathcal{A}, D)\) be a differential algebra, and let 
\(\phi: (\mathfrak{g}, [\cdot, \cdot], d) \rightarrow (\mathcal{A}, [\cdot, \cdot], D)\) be a homomorphism of differential Lie algebras.

Then, there is a unique homomorphism of differential algebras 
\(\hat{\phi}: (\mathcal{U}(\mathfrak{g}), \tilde{\partial}) \rightarrow (\mathcal{A}, D)\) such that \(\hat{\phi} \circ j = \phi\), where \(j: \mathfrak{g} \rightarrow \mathcal{U}(\mathfrak{g})\) is the canonical differential Lie map.
The universal enveloping algebra \textbf{lifts} to the realm of differential algebras. Hence symbolically one has

\[
\begin{array}{c}
(g, d) \\ \\
\downarrow \\
(g) \\
\end{array} \xrightarrow{j} \begin{array}{c}
(U(g), \tilde{\partial}) \\ \\
\downarrow \\
(U(g)) \\
\end{array}
\]
Conclusion for the first approach

The universal enveloping algebra lifts to the realm of differential algebras. Hence symbolically one has

\[(g, d) \rightarrow (\mathcal{U}(g), \tilde{\partial})\]

PBW theorem remains unchanged.
Let \((A, d)\) be a **commutative** differential (associative and unital) \(R\)-algebra.

Let us define the **Wronskian bracket**

\[
W(x, y) := xd(y) - d(x)y.
\]

Of course it is **alternating** \(W(x, x) = xd(x) - d(x)x = 0\) (since \(A\) is commutative).

Moreover it satisfies **Jacobi identity**.

Hence \((A, W)\) turns to be a Lie algebra.
Furthermore $d(W(x, y)) = d(xd(y) - d(x)y)$
Furthermore \( d(W(x, y)) = d(xd(y) - d(x)y) = d(x)d(y) + xd^2(y) - d^2(x)y - d(x)d(y) \)
Furthermore \( d(W(x, y)) = d(xd(y) - d(x)y) = \)
\( d(x)d(y) + xd^2(y) - d^2(x)y - d(x)d(y) = xd^2(y) - d^2(x)y. \)
Furthermore \( d(W(x, y)) = d(xd(y) - d(x)y) = d(x)d(y) + xd^2(y) - d^2(x)y - d(x)d(y) = xd^2(y) - d^2(x)y \).

While
\[
W(d(x), y) + W(x, d(y)) = d(x)d(y) - d^2(x)y + xd^2(y) - d(x)d(y).
\]

Hence \((A, W, d)\) is a differential Lie algebra.
Furthermore \( d(W(x, y)) = d(xd(y) - d(x)y) = d(x)d(y) + xd^2(y) - d^2(x)y - d(x)d(y) = xd^2(y) - d^2(x)y. \)

While
\[
W(d(x), y) + W(x, d(y)) = d(x)d(y) - d^2(x)y + xd^2(y) - d(x)d(y).
\]

Hence \((A, W, d)\) is a differential Lie algebra.

This defines a functor, say the Wronskian, \((A, d) \mapsto (A, W, d)\) from \(\text{DiffComAss}\) to \(\text{DiffLie}\).

**Remark**
Composing with the obvious forgetful functor \(\text{DiffLie} \to \text{Lie}\), the above construction provides a functor \((A, d) \mapsto (A, W)\) from \(\text{DiffComAss}\) to \(\text{Lie}\).
One observes that the Wronskian functor preserves the obvious forgetful functors,

so it is an algebraic functor,

and it admits a left adjoint $\mathcal{W}$, the Wronskian enveloping algebra.
Construction of the differential enveloping algebra (1/2)

1st step: universal extension of the derivation on the symmetric algebra

Let $(\mathfrak{g}, [\cdot, \cdot], d)$ be a differential Lie algebra.

Let $S(\mathfrak{g})$ be the symmetric algebra of the module $\mathfrak{g}$ which becomes a commutative differential algebra with the unique derivation $\partial$ that extends the map $\partial(x) = d(x)$ on the generators $x \in \mathfrak{g}$. 
Construction of the differential enveloping algebra (1/2)
1st step: universal extension of the derivation on the symmetric algebra

Let \((\mathfrak{g}, [-, -], d)\) be a differential Lie algebra.

Let \(S(\mathfrak{g})\) be the symmetric algebra of the module \(\mathfrak{g}\) which becomes a commutative differential algebra with the unique derivation \(\partial\) that extends the map \(\partial(x) = d(x)\) on the generators \(x \in \mathfrak{g}\).

**Remark**
Actually, one defines the derivation \(\partial\) on the tensor algebra \(T(\mathfrak{g})\), and since it commutes to the permutation of variables, it factors through \(S(\mathfrak{g})\).
Let us consider the (algebraic) ideal $I$ generated by $d(x)y - xd(y) - [x, y]$, $x, y \in g$. 
Let us consider the (algebraic) ideal $I$ generated by $d(x)y - xd(y) - [x, y]$, $x, y \in g$.

One observes that $\partial(I) \subseteq I$. Hence $I$ is actually a differential ideal.
Let us consider the (algebraic) ideal $I$ generated by $d(x)y - xd(y) - [x, y], x, y \in g$.

One observes that $\partial(I) \subseteq I$. Hence $I$ is actually a differential ideal.

Then, the Wronskian enveloping algebra $\mathcal{W}(g, [-,-], d)$ is $(S(g)/I, \tilde{\partial})$.
Universal property of the Wronskian enveloping algebra

Let \((A, \delta)\) be any commutative differential algebra, and let 
\(\phi: (g, [-, -], d) \mapsto (A, W, \delta)\) be a homomorphism of differential Lie algebras.
Universal property of the Wronskian enveloping algebra

Let \((A, \delta)\) be any commutative differential algebra, and let \(\phi: (g, [-,-], d) \mapsto (A, W, \delta)\) be a homomorphism of differential Lie algebras.

Then, there exists a unique differential algebra map
\(\tilde{\phi}: (S(g)/I, \tilde{\partial}) \to (A, \delta)\) such that \(\tilde{\phi}(x + I) = \phi(x)\) for each \(x \in g\).
Proof

Let \( \hat{\phi} : S(g) \to A \) be the unique algebra map that extends \( \phi \).
Proof

Let $\hat{\phi}: S(\mathfrak{g}) \to A$ be the unique algebra map that extends $\phi$.

One easily observes that $\hat{\phi}$ commutes to the derivations, and so defines a homomorphism of differential algebras.
Proof

Let $\hat{\phi}: S(\mathfrak{g}) \to A$ be the unique algebra map that extends $\phi$.

One easily observes that $\hat{\phi}$ commutes to the derivations, and so defines a homomorphism of differential algebras.

Moreover it satisfies
$$\hat{\phi}(d(x)y - xd(y) - [x, y]) = \delta(\phi(x))\phi(y) - \phi(x)\delta(\phi(y)) - [\phi(x), \phi(y)]$$
Proof

Let \( \hat{\phi} : S(\mathfrak{g}) \to A \) be the unique algebra map that extends \( \phi \).

One easily observes that \( \hat{\phi} \) commutes to the derivations, and so defines a homomorphism of differential algebras.

Moreover it satisfies
\[
\hat{\phi}(d(x)y - xd(y) - [x, y]) = \delta(\phi(x))\phi(y) - \phi(x)\delta(\phi(y)) - [\phi(x), \phi(y)] = W(\phi(x), \phi(y)) - [\phi(x), \phi(y)]
\]
Proof

Let $\hat{\phi}: S(\mathfrak{g}) \rightarrow A$ be the unique algebra map that extends $\phi$.

One easily observes that $\hat{\phi}$ commutes to the derivations, and so defines a homomorphism of differential algebras.

Moreover it satisfies

$$
\hat{\phi}(d(x)y - xd(y) - [x, y]) = \delta(\phi(x))\phi(y) - \phi(x)\delta(\phi(y)) - [\phi(x), \phi(y)] = W(\phi(x), \phi(y)) - [\phi(x), \phi(y)] = 0.
$$
Proof

Let \( \hat{\phi} : S(g) \to A \) be the unique algebra map that extends \( \phi \).

One easily observes that \( \hat{\phi} \) commutes to the derivations, and so defines a homomorphism of differential algebras.

Moreover it satisfies
\[
\hat{\phi}(d(x)y - xd(y) - [x, y]) = \delta(\phi(x))\phi(y) - \phi(x)\delta(\phi(y)) - [\phi(x), \phi(y)] = W(\phi(x), \phi(y)) - [\phi(x), \phi(y)] = 0.
\]

Hence it factors through \( I \) and provides a unique homomorphism of differential algebras \( \tilde{\phi} \) from \( (S(g)/I, \tilde{\partial}) \) to \( (A, \delta) \) such that
\[
\tilde{\phi}(x + I) = \phi(x), \; x \in g.
\]
Table of contents

1 Motivations

2 A glance at universal (differential) algebra

3 Differential Lie algebras and their enveloping differential algebras

4 The embedding problem

5 Jacobi, Poisson and Lie-Rinehart algebras

6 Kirillov’s local Lie algebras and Lie algebroids
Statement of the problem

Given a differential Lie $R$-algebra $(g, d)$, and its Wronskian enveloping algebra $(\mathcal{W}(g, d), \tilde{\partial})$, the (differential) Lie map \( \text{can}: g \rightarrow S(g)/I, \quad x \mapsto x + I, \) is referred to as the canonical map.
Statement of the problem

Given a differential Lie $R$-algebra $(g, d)$, and its Wronskian enveloping algebra $(\mathcal{W}(g, d), \bar{\partial})$, the (differential) Lie map $\text{can}: g \rightarrow S(g)/I$, $x \mapsto x + I$, is referred to as the canonical map.

Embedding problem

Under which conditions on $(g, d)$ and on $R$ is the canonical map one-to-one?
Statement of the problem

Given a differential Lie $R$-algebra $(g, d)$, and its Wronskian enveloping algebra $(\mathcal{W}(g, d), \tilde{\partial})$, the (differential) Lie map $\text{can}: g \rightarrow S(g)/I$, $x \mapsto x + I$, is referred to as the canonical map.

Embedding problem

Under which conditions on $(g, d)$ and on $R$ is the canonical map one-to-one?

Remark

$\text{can}$ is one-to-one if, and only if, there are a differential commutative algebra $(A, \delta)$, and a one-to-one differential Lie map $\phi: (g, d) \rightarrow ((A, W), \delta)$. 
Example: $\mathfrak{sl}_2(\mathbb{K})$

Let $\mathbb{K}$ be a field of characteristic zero.

The Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ embeds into the algebra of vector fields of $\mathbb{K}[x]$ by the identification of the elements of its Chevalley basis $e = -1$, $h = -2x$, and $f = x^2$ (the familiar commutation rules are satisfied $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$).

It is a differential Lie algebra when equipped with the usual derivation of polynomials.

Hence it embeds into the commutative differential algebra $(\mathbb{K}[x], \frac{d}{dx})$ as a sub-Lie algebra under the Wronskian bracket, therefore it embeds into its Wronskian enveloping algebra.
Warning: The case of a non-differential Lie algebra (1/3)

For Lie algebras without derivation, there are two different notions for the Wronskian envelope, depending on whether or not one identifies $\text{Lie}$ with a sub-category of $\text{DiffLie}$ via the embedding functor $g \mapsto (g, 0)$. Therefore, there are two formulations for the embedding problem.
Warning: The case of a non-differential Lie algebra (1/3)

For Lie algebras without derivation, there are two different notions for the Wronskian envelope, depending on whether or not one identifies $\text{Lie}$ with a sub-category of $\text{DiffLie}$ via the embedding functor $g \mapsto (g, 0)$.

- Hence, the Wronskian envelope of $g$ may be defined either as the Wronskian envelope $\mathcal{W}(g, 0)$ of the differential Lie algebra $(g, 0)$ with the zero derivation,
Warning: The case of a non-differential Lie algebra (1/3)

For Lie algebras without derivation, there are two different notions for the Wronskian envelope, depending on whether or not one identifies $\text{Lie}$ with a sub-category of $\text{DiffLie}$ via the embedding functor $g \mapsto (g, 0)$.

- Hence, the Wronskian envelope of $g$ may be defined either as the Wronskian envelope $\mathcal{W}(g, 0)$ of the differential Lie algebra $(g, 0)$ with the zero derivation,

- or it may be defined as a left adjoint to the composite forgetful functor $\text{DiffComAss} \overset{\text{Wronskian bracket}}{\longrightarrow} \text{DiffLie} \overset{\text{forgets der.}}{\longrightarrow} \text{Lie}$. 

Warning: The case of a non-differential Lie algebra (1/3)

For Lie algebras without derivation, there are two different notions for the Wronskian envelope, depending on whether or not one identifies $\mathbf{Lie}$ with a sub-category of $\text{DiffLie}$ via the embedding functor $g \mapsto (g, 0)$.

- Hence, the Wronskian envelope of $g$ may be defined either as the Wronskian envelope $\mathcal{W}(g, 0)$ of the differential Lie algebra $(g, 0)$ with the zero derivation,

- or it may be defined as a left adjoint to the composite forgetful functor $\text{DiffComAss} \xrightarrow{\text{Wronskian bracket}} \text{DiffLie} \xrightarrow{\text{forgets der.}} \mathbf{Lie}$.

Therefore, there are two formulations for the embedding problem.
Warning: The case of a non-differential Lie algebra (2/3)

As a Lie algebra with the zero derivation

Let \((\mathfrak{g}, [-, -])\) be a Lie algebra. It may be identified with the differential Lie algebra \((\mathfrak{g}, [-, -], 0)\).
Warning: The case of a non-differential Lie algebra (2/3)

As a Lie algebra with the zero derivation

Let \((\mathfrak{g}, [-, -])\) be a Lie algebra. It may be identified with the differential Lie algebra \((\mathfrak{g}, [-, -], 0)\).

The derivation on \(S(\mathfrak{g})\) that extends the zero derivation is also just the zero derivation.
Warning: The case of a non-differential Lie algebra (2/3)
As a Lie algebra with the zero derivation

Let \((\mathfrak{g}, [\cdot, \cdot])\) be a Lie algebra. It may be identified with the differential Lie algebra \((\mathfrak{g}, [\cdot, \cdot], 0)\).

The derivation on \(S(\mathfrak{g})\) that extends the zero derivation is also just the zero derivation.

The differential ideal \(I\) is equal to the (algebraic) ideal generated by \([x, y], x, y \in \mathfrak{g}\).

Hence it follows that in case \(\mathfrak{g}\) is not commutative (i.e., \([\cdot, \cdot]\) does not vanish identically), \(\mathfrak{g}\) does not embed into its universal enveloping differential (commutative) algebra \(\mathcal{W}(\mathfrak{g})\) even if \(R\) is a field!
Warning: The case of a non-differential Lie algebra (2/3)

As a Lie algebra with the zero derivation

Let \((\mathfrak{g}, [-,-])\) be a Lie algebra. It may be identified with the differential Lie algebra \((\mathfrak{g}, [-,-], 0)\).

The derivation on \(S(\mathfrak{g})\) that extends the zero derivation is also just the zero derivation.

The differential ideal \(I\) is equal to the (algebraic) ideal generated by \([x, y]\), \(x, y \in \mathfrak{g}\).

Hence it follows that in case \(\mathfrak{g}\) is not commutative (i.e., \([-,-]\) does not vanish identically), \(\mathfrak{g}\) does not embed into its universal enveloping differential (commutative) algebra \(\mathcal{W}(\mathfrak{g})\) even if \(R\) is a field!

In this case, the embedding problem is rather obvious (of course, any commutative Lie algebra embeds into its Wronskian envelope, which reduced to the symmetric algebra).
The composite forgetful functor

\[
\text{DiffComAss} \xrightarrow{\text{Wronskian bracket}} \text{DiffLie} \xrightarrow{\text{forgets der.}} \text{Lie}
\]

is an algebraic functor, hence admits a left adjoint.

Thus, by composition of left adjoints, the Wronskian envelope of a Lie algebra \( g \) may be defined as the Wronskian envelope \( W(DL(g)) \) of the free differential Lie algebra \( DL(g) \) generated by the Lie algebra \( g \).

Embedding problem

Under which conditions on \( g \) and on \( R \) is the canonical map from \( g \) to \( W(DL(g)) \) one-to-one?
The composite forgetful functor

\[
\text{DiffComAss} \xrightarrow{\text{Wronskian bracket}} \text{DiffLie} \xrightarrow{\text{forgets der.}} \text{Lie}
\]

is an algebraic functor, hence admits a left adjoint.

Thus, by composition of left adjoints, the Wronskian envelope of a Lie algebra \( \mathfrak{g} \) may be defined as the Wronskian envelope \( \mathcal{W}(\mathcal{D}\mathcal{L}(\mathfrak{g})) \) of the free differential Lie algebra \( \mathcal{D}\mathcal{L}(\mathfrak{g}) \) generated by the Lie algebra \( \mathfrak{g} \).
The composite forgetful functor
\[
\text{DiffComAss} \xrightarrow{\text{Wronskian bracket}} \text{DiffLie} \xrightarrow{\text{forgets der.}} \text{Lie}
\]
is an algebraic functor, hence admits a left adjoint.

Thus, by composition of left adjoints, the Wronskian envelope of a Lie algebra \( g \) may be defined as the the Wronskian envelope \( \mathcal{W}(\mathcal{D}L(g)) \) of the free differential Lie algebra \( \mathcal{D}L(g) \) generated by the Lie algebra \( g \).

**Embedding problem**

Under which conditions on \( g \) and on \( R \) is the canonical map from \( g \) to \( \mathcal{W}(\mathcal{D}L(g)) \) one-to-one?
Remark

The canonical map $g \rightarrow \mathcal{W}(DL(g))$ is one-to-one if, and only if, there are a differential commutative algebra $(A, \delta)$, and a one-to-one Lie map $\phi: g \rightarrow (A, W)$. 

Indeed, in this case there is a unique differential Lie algebra map $\hat{\phi}: (DL(g), d) \rightarrow ((A, W), \delta)$ such that $\hat{\phi} \circ \text{can} \_g = \phi$, where $\text{can} \_g: g \rightarrow DL(g)$ is the canonical map (a Lie algebra map).

Then, there is a unique differential algebra map $\hat{\hat{\phi}}: (W(DL(g)), d) \rightarrow (A, \delta)$ such that $\hat{\hat{\phi}} \circ \text{can} \_g = \hat{\phi}$, hence $\hat{\hat{\phi}} \circ \text{can} \_g \circ \text{can} \_g = \phi$ which implies that $\text{can} \_g \circ \text{can} \_g : g \rightarrow DL(g) \rightarrow W(DL(g))$ is one-to-one.
Remark

The canonical map $\mathfrak{g} \to \mathcal{W}(\mathcal{D}\mathcal{L}(\mathfrak{g}))$ is one-to-one if, and only if, there are a differential commutative algebra $(A, \delta)$, and a one-to-one Lie map $\phi: \mathfrak{g} \to (A, \mathcal{W})$.

Indeed, in this case there is a unique differential Lie algebra map $\hat{\phi}: (\mathcal{D}\mathcal{L}(\mathfrak{g}), d) \to ((A, \mathcal{W}), \delta)$ such that $\hat{\phi} \circ \text{can}_\mathfrak{g} = \phi$, where $\text{can}_\mathfrak{g}: \mathfrak{g} \to \mathcal{D}\mathcal{L}(\mathfrak{g})$ is the canonical map (a Lie algebra map).
Remark

The canonical map \( g \to \mathcal{W}(\mathcal{D}L(g)) \) is one-to-one if, and only if, there are a differential commutative algebra \((A, \delta)\), and a one-to-one Lie map \( \phi: g \to (A, W) \).

Indeed, in this case there is a unique differential Lie algebra map \( \hat{\phi}: (\mathcal{D}L(g), d) \to ((A, W), \delta) \) such that \( \hat{\phi} \circ \text{can}_g = \phi \), where \( \text{can}_g: g \to \mathcal{D}L(g) \) is the canonical map (a Lie algebra map).

Then, there is a unique differential algebra map \( \hat{\phi}: (\mathcal{W}(\mathcal{D}L(g)), d) \to (A, \delta) \) such that \( \hat{\phi} \circ \text{can} = \hat{\phi} \), hence

\[ \hat{\phi} \circ \text{can} \circ \text{can}_g = \phi \]

which implies that

\[ \text{can} \circ \text{can}_g = g \xrightarrow{\text{can}_g} \mathcal{D}L(g) \xrightarrow{\text{can}} \mathcal{W}(\mathcal{D}L(g)) \] is one-to-one.
Augmented modules

Let \((M, \epsilon)\) be an augmented \(R\)-module, i.e., a \(R\)-module together with a linear map \(\epsilon: M \rightarrow R\), called its augmentation map.
Augmented modules

Let \((M, \epsilon)\) be an augmented \(R\)-module, i.e., a \(R\)-module together with a linear map \(\epsilon : M \rightarrow R\), called its augmentation map.

It admits a Lie bracket

\[
[u, v]_\epsilon := \epsilon(v)u - \epsilon(u)v.
\]

The Lie algebra \((M, [-, -]_\epsilon)\) is referred to as the associated Lie algebra of \((M, \epsilon)\).
Let \((M, \epsilon)\) be an augmented \(R\)-module, i.e., a \(R\)-module together with a linear map \(\epsilon: M \to R\), called its augmentation map.

It admits a Lie bracket

\[
[u, v]_\epsilon := \epsilon(v)u - \epsilon(u)v.
\]

The Lie algebra \((M, [\cdot, \cdot]_\epsilon)\) is referred to as the associated Lie algebra of \((M, \epsilon)\).

**Proposition**

The associated Lie algebra of an augmented module embeds into its Wronskian envelope.
Sketch of the proof

Given an augmented module \((M, \epsilon)\), it can be shown that there is a unique derivation \(d_\epsilon\) on the symmetric algebra \(S(M)\) of \(M\) that extends \(\epsilon\).

Let \(u, v \in M\). Then, \(W(u, v) = ud_\epsilon(v) - d_\epsilon(u)v = u\epsilon(v) - \epsilon(u)v = [u, v]\). Hence the canonical embedding \(M \hookrightarrow S(M)\) is a Lie map.
Modules with a “rank one” projection

Let $M$ be a $R$-module. Let $P : M \rightarrow M$ be a rank one (linear) projection, i.e., $P^2 = P$ and $im(P) \simeq R$ (as modules).
Modules with a “rank one” projection

Let $M$ be a $R$-module. Let $P : M \to M$ be a rank one (linear) projection, i.e., $P^2 = P$ and $\text{im}(P) \cong R$ (as modules).

Remark

It is essentially the same object as an augmented module $(M, \epsilon)$ with a surjective augmentation map $\epsilon$, because in this case, since $R$ is free on $\{1\}$, the short exact sequence $0 \to \ker \epsilon \hookrightarrow M \xrightarrow{\epsilon} R \to 0$ splits, so $M \cong \ker \epsilon \oplus Re$ (with $\epsilon(e) = 1$), and one has a rank one projection $P(x) := \epsilon(x)e$. 
Modules with a “rank one” projection

Let $M$ be a $R$-module. Let $P : M \to M$ be a rank one (linear) projection, i.e., $P^2 = P$ and $im(P) \cong R$ (as modules).

Remark

It is essentially the same object as an augmented module $(M, \epsilon)$ with a surjective augmentation map $\epsilon$, because in this case, since $R$ is free on $\{1\}$, the short exact sequence $0 \to \ker \epsilon \hookrightarrow M \xrightarrow{\epsilon} R \to 0$ splits, so $M \cong \ker \epsilon \oplus Re$ (with $\epsilon(e) = 1$), and one has a rank one projection $P(x) := \epsilon(x)e$.

Conversely, if $P$ is a rank one projection on $M$, then for each $x \in M$ there is a unique scalar $\langle P(x) \mid e \rangle \in R$ such that $P(x) = \langle P(x) \mid e \rangle e$, where $e$ a generator of $im(P) \cong R$. Then, $\langle P(\cdot) \mid e \rangle : M \to R$ is a surjective augmentation map.
Modules with a “rank one” projection

Let $M$ be a $R$-module. Let $P : M \to M$ be a rank one (linear) projection, i.e., $P^2 = P$ and $\text{im}(P) \cong R$ (as modules).

Remark

It is essentially the same object as an augmented module $(M, \epsilon)$ with a surjective augmentation map $\epsilon$, because in this case, since $R$ is free on $\{1\}$, the short exact sequence $0 \to \ker \epsilon \hookrightarrow M \overset{\epsilon}{\to} R \to 0$ splits, so $M \cong \ker \epsilon \oplus Re$ (with $\epsilon(e) = 1$), and one has a rank one projection $P(x) := \epsilon(x)e$.

Conversely, if $P$ is a rank one projection on $M$, then for each $x \in M$ there is a unique scalar $\langle P(x) \mid e \rangle \in R$ such that $P(x) = \langle P(x) \mid e \rangle e$, where $e$ a generator of $\text{im}(P) \cong R$. Then, $\langle P(\cdot) \mid e \rangle : M \to R$ is a surjective augmentation map.

Once chosen a generator $e$ of $\text{im}(P)$, one has a Lie algebra structure on $M$ given by $[u, v] = \langle P(v) \mid e \rangle u - \langle P(u) \mid e \rangle v$. 

An application (1/2)

Let $(A, d)$ be a commutative $R$-algebra with a unit.
Let $(A, d)$ be a commutative $R$-algebra with a unit.

Let $A^d := \{ a \in A : d(a) = 0 \} = \ker d$ be the ring of constants of $(A, d)$ (it is even a $R$-sub-algebra of $A$, and $(A, d) \mapsto A^d$ is a functorial correspondence).
An application (1/2)

Let \((A, d)\) be a commutative \(R\)-algebra with a unit.

Let \(A^d := \{ a \in A: d(a) = 0 \} = \ker d\) be the ring of constants of \((A, d)\) (it is even a \(R\)-sub-algebra of \(A\), and \((A, d) \mapsto A^d\) is a functorial correspondence).

Let \(\text{Fix}(A, d) := \{ a \in A: d(a) = a \}\) be the \(R\)-module of fixed points of \(d\). (Again, \((A, d) \mapsto \text{Fix}(A, d)\) is functorial.)
Let \((A, d)\) be a commutative \(R\)-algebra with a unit.

Let \(A^d := \{ a \in A : d(a) = 0 \} = \ker d\) be the ring of constants of \((A, d)\) (it is even a \(R\)-sub-algebra of \(A\), and \((A, d) \mapsto A^d\) is a functorial correspondence).

Let \(\text{Fix}(A, d) := \{ a \in A : d(a) = a \}\) be the \(R\)-module of fixed points of \(d\). (Again, \((A, d) \mapsto \text{Fix}(A, d)\) is functorial.)

One has \(A^d \cap \text{Fix}(A, d) = (0)\), and \(A^d \oplus \text{Fix}(A, d) = \{ a \in A : d^2(a) = d(a) \}\).
An application (1/2)

Let \((A, d)\) be a commutative \(R\)-algebra with a unit.

Let \(A^d := \{ a \in A : d(a) = 0 \} = \ker d\) be the ring of constants of \((A, d)\)
(it is even a \(R\)-sub-algebra of \(A\), and \((A, d) \mapsto A^d\) is a functorial
Correspondence).

Let \(\text{Fix}(A, d) := \{ a \in A : d(a) = a \}\) be the \(R\)-module of fixed points of \(d\).
(Again, \((A, d) \mapsto \text{Fix}(A, d)\) is functorial.)

One has \(A^d \cap \text{Fix}(A, d) = (0)\), and
\(A^d \oplus \text{Fix}(A, d) = \{ a \in A : d^2(a) = d(a) \}\).

Moreover, the restriction of \(d\) to \(A^d \oplus \text{Fix}(A, d)\) is a linear projection with
\(\text{im}(d) = \text{Fix}(A, d)\).
An application (2/2)

Assuming that the ring of constants $A^d$ is $R_{1A} \simeq R$, one gets a rank one projection $id - d$ on $R_{1A} \oplus Fix(A, d)$ onto $R_{1A}$. 
Assuming that the ring of constants $A^d$ is $R_1A \cong R$, one gets a rank one projection $id - d$ on $R_1A \oplus \text{Fix}(A, d)$ onto $R_1A$.

The associated Lie bracket is thus given by

$$[x, y] = \langle y - d(y) | 1_A \rangle x - \langle x - d(x) | 1_A \rangle y.$$
An application (2/2)

Assuming that the ring of constants $A^d$ is $R_{1A} \cong R$, one gets a rank one projection $id - d$ on $R_{1A} \oplus \text{Fix}(A, d)$ onto $R_{1A}$.

The associated Lie bracket is thus given by

$$[x, y] = \langle y - d(y) | 1_A \rangle x - \langle x - d(x) | 1_A \rangle y.$$ 

Example

1. Let $R[x]$ with its usual derivation $d(x) = 1$. Then, $R[x]^d = R$ and $\text{Fix}(R[x], d) = (0)$. Then, $[r, s] = 0$ for all $r, s \in R$. 

An application (2/2)

Assuming that the ring of constants $A^d$ is $R1_A \simeq R$, one gets a rank one projection $id - d$ on $R1_A \oplus \text{Fix}(A, d)$ onto $R1_A$.

The associated Lie bracket is thus given by
\[
[x, y] = \langle y - d(y) | 1_A \rangle x - \langle x - d(x) | 1_A \rangle y.
\]

Example

1. Let $R[x]$ with its usual derivation $d(x) = 1$. Then, $R[x]^d = R$ and $\text{Fix}(R[x], d) = (0)$. Then, $[r, s] = 0$ for all $r, s \in R$.

2. Let $d$ be the unique derivation of $R[x]$ such that $d(x) = x$. Then, $d(x^n) = nx^n$. It follows that $\text{Fix}(R[x], d) = Rx$ and $R[x]^d = R$. Hence
\[
[x + s, tx + v] = (rx + s)(tx + v - d(tx + v)) | 1\rangle - (tx + v)(rx + s - d(rx + s)) | 1\rangle = (rx + s)v - (tx + v)s = x(rv - st).
\]
Remark: Change of base ring

Let \((A, d)\) be a commutative differential algebra. Then \((A, d)\) is also a \(A^d\)-algebra, and \(d\) is a \(A^d\)-derivation.
Remark: Change of base ring

Let \((A, d)\) be a commutative differential algebra. Then \((A, d)\) is also a \(A^d\)-algebra, and \(d\) is a \(A^d\)-derivation. Hence \((A, d)\) becomes a commutative differential \(A^d\)-algebra denoted by \((A, d)_{A^d}\).
Remark: Change of base ring

Let \((A, d)\) be a commutative differential algebra. Then \((A, d)\) is also a \(A^d\)-algebra, and \(d\) is a \(A^d\)-derivation. Hence \((A, d)\) becomes a commutative differential \(A^d\)-algebra denoted by \((A, d)_{A^d}\).

Moreover, \((A, d)_{A^d}^d = A^d\), and \(\text{Fix}((A, d)_{A^d}) = \text{Fix}(A, d)\) (as \(A^d\)-modules).
Remark: Change of base ring

Let $(A, d)$ be a commutative differential algebra. Then $(A, d)$ is also a $A^d$-algebra, and $d$ is a $A^d$-derivation. Hence $(A, d)$ becomes a commutative differential $A^d$-algebra denoted by $(A, d)_A^d$.

Moreover, $(A, d)^d_A = A^d$, and $\text{Fix}((A, d)_A^d) = \text{Fix}(A, d)$ (as $A^d$-modules).

Therefore, the previous construction applies, and $A^d \oplus (A, d)$ turns to be a Lie $A^d$-algebra that embeds into its Wronskian envelope (via the canonical Lie map).
Table of contents

1 Motivations

2 A glance at universal (differential) algebra

3 Differential Lie algebras and their enveloping differential algebras

4 The embedding problem

5 Jacobi, Poisson and Lie-Rinehart algebras

6 Kirillov’s local Lie algebras and Lie algebroids
Lie-Rinehart algebras

Let \((M, \cdot)\) be a (not necessarily associative) \(R\)-algebra. Let \(\mathcal{Der}_R(M, \cdot)\) be its Lie \(R\)-algebra of \(R\)-linear derivations (under the usual commutator bracket). When \((M, \cdot)\) is commutative, \(\mathcal{Der}_R(M, \cdot)\) becomes a \(A\)-module in an obvious way.
Lie-Rinehart algebras

Let \((M, \cdot)\) be a (not necessarily associative) \(R\)-algebra. Let \(\text{Der}_R(M, \cdot)\) be its Lie \(R\)-algebra of \(R\)-linear derivations (under the usual commutator bracket). When \((M, \cdot)\) is commutative, \(\text{Der}_R(M, \cdot)\) becomes a \(A\)-module in an obvious way.

**Definition**

A **Lie-Rinehart algebra** over \(R\) is a triple \((A, g, \delta)\), where

- \(A\) is a commutative \(R\)-algebra with a unit,
- \(g\) is a Lie \(R\)-algebra which is also a left \(A\)-module (with \(A\)-action denoted by \(a \cdot x\)),
- \(\delta : g \rightarrow \text{Der}_R(A)\) is both a Lie \(R\)-algebra map, and a \(A\)-linear map \((\delta(a \cdot x)(b) = a(\delta(x)(b)))\) which turns \(A\) into a \(g\)-module,
- \([x, a \cdot y] = a \cdot [x, y] + \delta(x)(a) \cdot y, \ a \in A, \ x, y \in g\).
Lie-Rinehart algebras

Let \((M, \cdot)\) be a (not necessarily associative) \(R\)-algebra. Let \(\mathcal{Der}_R(M, \cdot)\) be its Lie \(R\)-algebra of \(R\)-linear derivations (under the usual commutator bracket). When \((M, \cdot)\) is commutative, \(\mathcal{Der}_R(M, \cdot)\) becomes a \(A\)-module in an obvious way.

Definition

A Lie-Rinehart algebra over \(R\) is a triple \((A, \mathfrak{g}, \vartheta)\), where

- \(A\) is a commutative \(R\)-algebra with a unit,
- \(\mathfrak{g}\) is a Lie \(R\)-algebra which is also a left \(A\)-module (with \(A\)-action denoted by \(a \cdot x\)),
- \(\vartheta : \mathfrak{g} \to \mathcal{Der}_R(A)\) is both a Lie \(R\)-algebra map, and a \(A\)-linear map \((\vartheta(a \cdot x)(b) = a(\vartheta(x)(b)))\) which turns \(A\) into a \(\mathfrak{g}\)-module,
- \([x, a \cdot y] = a \cdot [x, y] + \vartheta(x)(a) \cdot y, \ a \in A, \ x, y \in \mathfrak{g}\.\)

By abuse, \(\vartheta\) is referred to as the anchor map of the Lie-Rinehart algebra \((A, \mathfrak{g})\).
Remark and example

The structure of a Lie-Rinehart algebra is modeled on the properties of the pair \((C^\infty(V), \mathfrak{X}(V))\), where \(V\) is a finite-dimensional smooth manifold, \(C^\infty(V)\) is the ring of smooth functions on \(V\), and \(\mathfrak{X}(V)\) is the Lie algebra of smooth vector fields on \(V\).
Remark and example

The structure of a Lie-Rinehart algebra is modeled on the properties of the pair \((C^\infty(V), \mathfrak{X}(V))\), where \(V\) is a finite-dimensional smooth manifold, \(C^\infty(V)\) is the ring of smooth functions on \(V\), and \(\mathfrak{X}(V)\) is the Lie algebra of smooth vector fields on \(V\).

Example

Let \(A\) be a commutative \(R\)-algebra with a unit. Then, \((A, \text{Der}_R(A))\) is a Lie-Rinehart algebra.
Remark and example

The structure of a Lie-Rinehart algebra is modeled on the properties of the pair \((C^\infty(V), \mathfrak{X}(V))\), where \(V\) is a finite-dimensional smooth manifold, \(C^\infty(V)\) is the ring of smooth functions on \(V\), and \(\mathfrak{X}(V)\) is the Lie algebra of smooth vector fields on \(V\).

Example

Let \(A\) be a commutative \(R\)-algebra with a unit. Then, \((A, \text{Der}_R(A))\) is a Lie-Rinehart algebra.

Given a Lie-Rinehart algebra \((A, \mathfrak{g})\), the Lie algebra \(\mathfrak{g}\), together with the anchor, is also referred to as a Lie \((R, A)\)-pseudoalgebra.
Any commutative differential $R$-algebra $(A, d)$ may be turned into a Lie-Rinehart algebra $(A, (A, W))$ with anchor map $a \mapsto \partial(a) := ad$, and this is functorial. This allows to view $\text{DiffComAss}$ as sub-category of $\text{LieRin}$. 
Any commutative differential $R$-algebra $(A, d)$ may be turned into a Lie-Rinehart algebra $(A, (A, W))$ with anchor map $a \mapsto \partial(a) := ad$, and this is functorial. This allows to view $\text{DiffComAss}$ as sub-category of $\text{LieRin}$. In particular, any commutative $R$-algebra $A$ provides a Lie-Rinehart algebra $(A, (A, 0))$. 
Some functors

Any commutative differential $R$-algebra $(A, d)$ may be turned into a Lie-Rinehart algebra $(A, (A, W))$ with anchor map $a \mapsto \partial(a) := ad$, and this is functorial. This allows to view $\text{DiffComAss}$ as sub-category of $\text{LieRin}$.

In particular, any commutative $R$-algebra $A$ provides a Lie-Rinehart algebra $(A, (A, 0))$.

It also provides another Lie-Rinehart algebra, namely $(A, (0))$, which is even the free Lie-Rinehart algebra generated by $A$. 
Any commutative differential $R$-algebra $(A, d)$ may be turned into a Lie-Rinehart algebra $(A, (A, W))$ with anchor map $a \mapsto \partial(a) := ad$, and this is functorial. This allows to view $\text{DiffComAss}$ as sub-category of $\text{LieRin}$.

In particular, any commutative $R$-algebra $A$ provides a Lie-Rinehart algebra $(A, (A, 0))$.

It also provides another Lie-Rinehart algebra, namely $(A, (0))$, which is even the free Lie-Rinehart algebra generated by $A$.

There is also a forgetful functor $\text{LieRin} \to \text{Lie}$, and it admits a left adjoint given on objects by $\mathfrak{g} \mapsto (R, \mathfrak{g})$. (This may also be interpreted as an embedding of $\text{Lie}$ into the category of Lie $(R, R)$-pseudoalgebras.)
Wronskian envelope of a Lie-Rinehart algebra (sketch)

\textbf{DiffComAss} is a reflective sub-category of \textbf{LieRin}.
**Wronskian envelope of a Lie-Rinehart algebra (sketch)**

**DiffComAss** is a reflective sub-category of **LieRin**.

Let \((A, g)\) be a Lie-Rinehart algebra with anchor map \(\partial\). Let \(\mathcal{D}(A, g)\) be the free commutative differential \(R\)-algebra generated by the set \(|A| \sqcup |g|\). Hence it is the commutative algebra of differential polynomials \(R\{ |A| \sqcup |g| \}\) with variables in \(|A| \sqcup |g|\).
Wronskian envelope of a Lie-Rinehart algebra (sketch)

DiffComAss is a reflective sub-category of LieRin.

Let $(A, g)$ be a Lie-Rinehart algebra with anchor map $\partial$. Let $\mathcal{D}(A, g)$ be the free commutative differential $R$-algebra generated by the set $|A| \sqcup |g|$. Hence it is the commutative algebra of differential polynomials $R\{ |A| \sqcup |g| \}$ with variables in $|A| \sqcup |g|$.

Then, let $I(A, g)$ be the differential ideal of $\mathcal{D}(A, g)$ generated by the relations that turn the canonical map $(A, g) \to (\mathcal{D}(A, g), (\mathcal{D}(A, g), W))$ into a Lie-Rinehart map. Then, $\mathcal{D}(A, g)/I(A, g)$ is the free commutative differential algebra generated by $(A, g)$.
A Jacobi algebra is a commutative $R$-algebra with a unit, together with a Lie bracket (called a Jacobi bracket) over $R$ which satisfies Jacobi-Leibniz rule:

$$[ab, c] = a[b, c] + b[a, c] - ab[1_A, c]$$

$a, b, c \in A$. 

It follows that $ad_{1_A} : A \rightarrow A$ is a $R$-derivation of the associative algebra $A$, and that $[-, -]_{W[1_A, -]}$ is an alternating biderivation.
A **Jacobi algebra** is a commutative $R$-algebra with a unit, together with a Lie bracket (called a **Jacobi bracket**) over $R$ which satisfies **Jacobi-Leibniz rule**:

$$[ab, c] = a[b, c] + b[a, c] − ab[1_A, c]$$

$a, b, c ∈ A$.

It follows that $ad_{1_A} = [1_A, ·]: A → A$ is a $R$-derivation of the associative algebra $A$, and that $[−, −] − W_{[1_A, −]}$ is an **alternating biderivation**.
Remark

Actually each triple \((A, D, d)\) where \(A\) is a commutative algebra, \(D\) is an alternating biderivation, and \(d\) is a derivation such that \(D + W_d\) is a Lie bracket provides a Jacobi algebra \((A, D + W_d)\).
### Remark

Actually each triple \((A, D, d)\) where \(A\) is a commutative algebra, \(D\) is an alternating biderivation, and \(d\) is a derivation such that \(D + W_d\) is a Lie bracket provides a Jacobi algebra \((A, D + W_d)\).

A commutative *Poisson algebra* thus is a Jacobi algebra whose associated derivation is zero.
### Remark

Actually each triple \((A, D, d)\) where \(A\) is a commutative algebra, \(D\) is an alternating biderivation, and \(d\) is a derivation such that \(D + W_d\) is a Lie bracket provides a Jacobi algebra \((A, D + W_d)\).

A commutative **Poisson algebra** thus is a Jacobi algebra whose associated derivation is zero.

A commutative differential algebra, with its Wronskian bracket, is a Jacobi algebra whose associated biderivation is zero.
Remark

Actually each triple \((A, D, d)\) where \(A\) is a commutative algebra, \(D\) is an alternating biderivation, and \(d\) is a derivation such that \(D + W_d\) is a Lie bracket provides a Jacobi algebra \((A, D + W_d)\).

A commutative Poisson algebra thus is a Jacobi algebra whose associated derivation is zero.

A commutative differential algebra, with its Wronskian bracket, is a Jacobi algebra whose associated biderivation is zero.

This provides two embedding functors

\[
\text{PoissCom} \leftrightarrow \text{Jac} \leftrightarrow \text{DiffComAss}.
\]
Remark
Actually each triple \((A, D, d)\) where \(A\) is a commutative algebra, \(D\) is an alternating biderivation, and \(d\) is a derivation such that \(D + W_d\) is a Lie bracket provides a Jacobi algebra \((A, D + W_d)\).

A commutative Poisson algebra thus is a Jacobi algebra whose associated derivation is zero.

A commutative differential algebra, with its Wronskian bracket, is a Jacobi algebra whose associated biderivation is zero.

This provides two embedding functors

\[
\text{PoissCom} \hookrightarrow \text{Jac} \hookrightarrow \text{DiffComAss}.
\]

Moreover, there is also a forgetful functor \(\text{Jac} \to \text{DiffComAss}, (A, [−, −]) \mapsto (A, [1_A, −])\).
Various envelopes

**PoissCom** is **reflective** in **Jac**: 
PoissCom is reflective in Jac: given a Jacobi algebra $(A, [\cdot, \cdot])$, let us consider its Jacobi ideal $I_{\text{poiss}}$ generated by $[1_A, x]$, $x \in A$, then $A/I_{\text{poiss}}$ is the free commutative Poisson algebra generated by $(A, [\cdot, \cdot])$. 
Various envelopes

PoissCom is reflective in Jac: given a Jacobi algebra \((A, [-, -])\), let us consider its Jacobi ideal \(I_{\text{poiss}}\) generated by \([1_A, x], x \in A\), then \(A/I_{\text{poiss}}\) is the free commutative Poisson algebra generated by \((A, [-, -])\).

DiffComAss is reflective in Jac, since the embedding functor is an algebraic functor between (equational) varieties.
Various envelopes

**PoissCom** is reflective in **Jac**: given a Jacobi algebra \((A, [−, −])\), let us consider its Jacobi ideal \(I_{\text{poiss}}\) generated by \([1_A, x], x \in A\), then \(A/I_{\text{poiss}}\) is the free commutative Poisson algebra generated by \((A, [−, −])\).

**DiffComAss** is reflective in **Jac**, since the embedding functor is an algebraic functor between (equational) varieties.

There is also a notion of a Jacobi envelope of a differential commutative algebra since the functor **Jac** \(→\) **DiffComAss** is an algebraic functor. One observes that any differential commutative algebra embeds into its Jacobi envelope.
Various envelopes

**PoissCom** is reflective in **Jac**: given a Jacobi algebra \((A, [-, -])\), let us consider its Jacobi ideal \(I_{\text{poiss}}\) generated by \([1_A, x]\), \(x \in A\), then \(A/I_{\text{poiss}}\) is the free commutative Poisson algebra generated by \((A, [-, -])\).

**DiffComAss** is reflective in **Jac**, since the embedding functor is an algebraic functor between (equational) varieties.

There is also a notion of a Jacobi envelope of a differential commutative algebra since the functor **Jac** \(\rightarrow\) **DiffComAss** is an algebraic functor. One observes that any differential commutative algebra embeds into its Jacobi envelope.

One finally mentions the composite forgetful functor **Jac** \(\rightarrow\) **DiffComAss** \(\rightarrow\) **LieRin**, \((A, [-, -]) \mapsto (A, (A, W_{ad_1A}))\), which makes it possible to consider the Jacobi envelope of a Lie-Rinehart algebra as the Jacobi envelope of the free commutative differential algebra generated by a Lie-Rinehart algebra.
Table of contents

1 Motivations

2 A glance at universal (differential) algebra

3 Differential Lie algebras and their enveloping differential algebras

4 The embedding problem

5 Jacobi, Poisson and Lie-Rinehart algebras

6 Kirillov’s local Lie algebras and Lie algebroids
The Lie side

There is also an obvious forgetful functor \( \text{Jac} \to \text{Lie} \), forgetting the multiplicative structure. It is an algebraic functor, so that it admits a left adjoint.
The Lie side

There is also an obvious forgetful functor $\text{Jac} \to \text{Lie}$, forgetting the multiplicative structure. It is an algebraic functor, so that it admits a left adjoint.

Given a Lie algebra $\mathfrak{g}$, one considers the free Jacobi algebra $\text{Jac}(\mathfrak{g})$ generated by the carrier set of $\mathfrak{g}$. 

There is also an obvious forgetful functor $\text{Jac} \to \text{Lie}$, forgetting the multiplicative structure. It is an algebraic functor, so that it admits a left adjoint.

Given a Lie algebra $\mathfrak{g}$, one considers the free Jacobi algebra $\text{Jac}(\|\mathfrak{g}\|)$ generated by the carrier set of $\mathfrak{g}$.

Let $J$ be its Jacobi ideal generated by the relations that make the canonical image of $\mathfrak{g}$ in $\text{Jac}(\|\mathfrak{g}\|)$ a Lie algebra.
There is also an obvious forgetful functor $\text{Jac} \to \text{Lie}$, forgetting the multiplicative structure. It is an algebraic functor, so that it admits a left adjoint.

Given a Lie algebra $\mathfrak{g}$, one considers the free Jacobi algebra $\text{Jac}(\|\mathfrak{g}\|)$ generated by the carrier set of $\mathfrak{g}$.

Let $J$ be its Jacobi ideal generated by the relations that make the canonical image of $\mathfrak{g}$ in $\text{Jac}(\|\mathfrak{g}\|)$ a Lie algebra.

Then, $\text{Jac}(\|\mathfrak{g}\|)/J$ is the universal Jacobi envelope of $\mathfrak{g}$. 

Relations between some envelopes

Because the following diagram of forgetful functors commutes, the
Wronskian envelope of a Lie algebra $\mathfrak{g}$ may be described as the free
differential commutative algebra generated by the Jacobi envelope of $\mathfrak{g}$.

\[
\begin{array}{c}
\text{Jac} & \xrightarrow{\text{DiffComAss}} & \text{Lie} \\
\end{array}
\]
Relations between some envelopes

Because the following diagram of forgetful functors commutes, the Wronskian envelope of a Lie algebra $\mathfrak{g}$ may be described as the free differential commutative algebra generated by the Jacobi envelope of $\mathfrak{g}$.

Moreover the following diagram of functors also commutes, implying that the Wronskian envelope of a Lie algebra $\mathfrak{g}$ is also the differential envelope of the Lie-Rinehart algebra $(R, \mathfrak{g})$. 
Local Lie algebras

Let $V$ be a finite-dimensional smooth manifold. Let $E$ be a line bundle over $V$, i.e., a vector bundle over $V$ each fibre of which is one-dimensional.

Let $\text{Sec}(E)$ be its space of global sections. $E$ is said to be trivial whenever $E = V \times \mathbb{R}$ in which case $C^\infty(V) = \text{Sec}(E)$.

Following A. A. Kirillov (1976), a local Lie algebra is a structure of a Lie algebra on $\text{Sec}(E)$ which is local, i.e., the support of $[s_1, s_2]$ is contained in the intersection of the supports of $s_1$ and $s_2$ (recall that the support of a section is the closure of the set of points at which the section does not vanish).

When $E$ is a trivial line bundle, then the local Lie bracket is of the form $[s_1, s_2] = \Lambda(ds_1, ds_2) + s_1 \Gamma(s_2) - \Gamma(s_1) s_2$ where $\Lambda$ is a bivector field, and $\Gamma$ is a vector field.

This implies that such a local Lie algebra $(C^\infty(V), [-,-])$ is precisely a Jacobi algebra.
Local Lie algebras

Let $V$ be a finite-dimensional smooth manifold. Let $E$ be a line bundle over $V$, i.e., a vector bundle over $V$ each fibre of which is one-dimensional.

Let Sec$(E)$ its space of global sections. $E$ is said to be trivial whenever $E = V \times \mathbb{R}$ in which case $C^\infty(V) = \text{Sec}(E)$.
Local Lie algebras

Let $V$ be a finite-dimensional smooth manifold. Let $E$ be a line bundle over $V$, i.e., a vector bundle over $V$ each fibre of which is one-dimensional.

Let $\text{Sec}(E)$ its space of global sections. $E$ is said to be trivial whenever $E = V \times \mathbb{R}$ in which case $C^\infty(V) = \text{Sec}(E)$.

Following A. A. Kirillov (1976), a local Lie algebra is a structure of a Lie algebra on $\text{Sec}(E)$ which is local, i.e., the support of $[s_1, s_2]$ is contained in the intersection of the supports of $s_1$ and $s_2$ (recall that the support of a section is the closure of the set of points at which the section does not vanish).
Local Lie algebras

Let $V$ be a finite-dimensional smooth manifold. Let $E$ be a line bundle over $V$, i.e., a vector bundle over $V$ each fibre of which is one-dimensional.

Let $\text{Sec}(E)$ its space of global sections. $E$ is said to be **trivial** whenever $E = V \times \mathbb{R}$ in which case $C^\infty(V) = \text{Sec}(E)$.

Following A. A. Kirillov (1976), a **local Lie algebra** is a structure of a Lie algebra on $\text{Sec}(E)$ which is **local**, i.e., the support of $[s_1, s_2]$ is contained in the intersection of the supports of $s_1$ and $s_2$ (recall that the support of a section is the closure of the set of points at which the section does not vanish).

When $E$ is a trivial line bundle, then the local Lie bracket is of the form

$$[s_1, s_2] = \Lambda(ds_1, ds_2) + s_1 \Gamma(s_2) - \Gamma(s_1)s_2$$

where $\Lambda$ is a bivector field, and $\Gamma$ is a vector field.
Local Lie algebras

Let $V$ be a finite-dimensional smooth manifold. Let $E$ be a line bundle over $V$, i.e., a vector bundle over $V$ each fibre of which is one-dimensional.

Let $\text{Sec}(E)$ its space of global sections. $E$ is said to be trivial whenever $E = V \times \mathbb{R}$ in which case $C^\infty(V) = \text{Sec}(E)$.

Following A. A. Kirillov (1976), a local Lie algebra is a structure of a Lie algebra on $\text{Sec}(E)$ which is local, i.e., the support of $[s_1, s_2]$ is contained in the intersection of the supports of $s_1$ and $s_2$ (recall that the support of a section is the closure of the set of points at which the section does not vanish).

When $E$ is a trivial line bundle, then the local Lie bracket is of the form

$$[s_1, s_2] = \Lambda(ds_1, ds_2) + s_1 \Gamma(s_2) - \Gamma(s_1)s_2$$

where $\Lambda$ is a bivector field, and $\Gamma$ is a vector field.

This implies that such a local Lie algebra $(C^\infty(V), [−, −])$ is precisely a Jacobi algebra.
Lie algebroids (1/2)

A Lie algebroid on a vector bundle $E$ over a finite-dimensional smooth manifold $V$ is a $(\mathbb{R}, C^\infty(V))$-Lie pseudoalgebra on the $C^\infty(V)$-module $\text{Sec}(E)$.

Example 1

A Lie algebroid on the tangent bundle $T^V$ is given by the canonical bracket $[\cdot, \cdot]_{vf}$ on $X(V) = \text{Sec}(TM)$.

Example 2

Every Lie algebra is a Lie algebroid over the one point manifold.
Lie algebroids (1/2)

A Lie algebroid on a vector bundle $E$ over a finite-dimensional smooth manifold $V$ is a $(\mathbb{R}, C^\infty(V))$-Lie pseudoalgebra on the $C^\infty(V)$-module $\text{Sec}(E)$.

The anchor map of the corresponding Lie-Rinehart algebra $(C^\infty(V), \text{Sec}(E))$ is described by a vector bundle morphism $d: E \to TV$ which induces the Lie map from $(\text{Sec}(E), [−, −])$ to the Lie algebra $(\mathcal{X}(V), [−, −]_{vf})$ of vector fields on $V$. 

Example 1

A Lie algebroid on the tangent bundle $TV$ is given by the canonical bracket $[−, −]_{vf}$ on $X(V) = \text{Sec}(TM)$.

Example 2

Every Lie algebra is a Lie algebroid over the one point manifold.
Lie algebroids (1/2)

A Lie algebroid on a vector bundle $E$ over a finite-dimensional smooth manifold $V$ is a $(\mathbb{R}, C^\infty(V))$-Lie pseudoalgebra on the $C^\infty(V)$-module $\text{Sec}(E)$.

The anchor map of the corresponding Lie-Rinehart algebra $(C^\infty(V), \text{Sec}(E))$ is described by a vector bundle morphism $d: E \to TV$ which induces the Lie map from $(\text{Sec}(E), [−, −])$ to the Lie algebra $(\mathfrak{X}(V), [−, −]_{\text{vf}})$ of vector fields on $V$.

Lie algebroids, introduced by J. Pradines (1967), are the infinitesimal parts of differentiable groupoids.
Lie algebroids (1/2)

A Lie algebroid on a vector bundle $E$ over a finite-dimensional smooth manifold $V$ is a $(\mathbb{R}, C^\infty(V))$-Lie pseudoalgebra on the $C^\infty(V)$-module $\text{Sec}(E)$.

The anchor map of the corresponding Lie-Rinehart algebra $(C^\infty(V), \text{Sec}(E))$ is described by a vector bundle morphism $d : E \to TV$ which induces the Lie map from $(\text{Sec}(E), [-,-])$ to the Lie algebra $(\mathfrak{X}(V), [-,-]_{vf})$ of vector fields on $V$.

Lie algebroids, introduced by J. Pradines (1967), are the infinitesimal parts of differentiable groupoids.

Example

1. A Lie algebroid on the tangent bundle $TV$ is given by the canonical bracket $[-,-]_{vf}$ on $\mathfrak{X}(V) = \text{Sec}(TM)$. 
A Lie algebroid on a vector bundle $E$ over a finite-dimensional smooth manifold $V$ is a $(\mathbb{R}, C^\infty(V))$-Lie pseudoalgebra on the $C^\infty(V)$-module $\text{Sec}(E)$.

The anchor map of the corresponding Lie-Rinehart algebra $(C^\infty(V), \text{Sec}(E))$ is described by a vector bundle morphism $d: E \to TV$ which induces the Lie map from $(\text{Sec}(E), [−, −])$ to the Lie algebra $(\mathfrak{X}(V), [−, −]_{vf})$ of vector fields on $V$.

Lie algebroids, introduced by J. Pradines (1967), are the infinitesimal parts of differentiable groupoids.

**Example**

1. A Lie algebroid on the tangent bundle $TV$ is given by the canonical bracket $[−, −]_{vf}$ on $\mathfrak{X}(V) = \text{Sec}(TM)$.

2. Every Lie algebra is a Lie algebroid over the one point manifold.
Lie algebroids (2/2)

Lie algebroids on the trivial line bundle, hence Lie algebroids brackets on $C^\infty(V)$, are particular local Lie algebras of the form

$$[f, g] = f\Gamma(g) - \Gamma(f)g$$

for a certain vector field $\Gamma$ on $V$. 

Remark

Other examples of embedding of a Lie pseudoalgebra into its Wronskian envelope are given by Lie algebras of vector fields tangent to a given foliation with one-dimensional leaves.
Lie algebroids (2/2)

Lie algebroids on the trivial line bundle, hence Lie algebroids brackets on $C^\infty(V)$, are particular local Lie algebras of the form

$$[f, g] = f\Gamma(g) - \Gamma(f)g$$

for a certain vector field $\Gamma$ on $V$.

It follows that the underlying Lie algebra of the Lie pseudoalgebra $(C^\infty(V), [−, −])$ embeds into its Wronskian envelope.
Lie algebroids (2/2)

Lie algebroids on the trivial line bundle, hence Lie algebroids brackets on $C^\infty(V)$, are particular local Lie algebras of the form

$$[f, g] = f \Gamma(g) - \Gamma(f)g$$

for a certain vector field $\Gamma$ on $V$.

It follows that the underlying Lie algebra of the Lie pseudoalgebra $(C^\infty(V), [-, -])$ embeds into its Wronskian envelope.

**Remark**

Other examples of embedding of a Lie pseudoalgebra into its Wronskian envelope are given by Lie algebras of vector fields tangent to a given foliation with one-dimensional leaves.
Conclusion

The embedding problem of a (differential) Lie algebra into its Wronskian enveloping algebra seems to be quite harder than the classical situation and related to Lie algebras of (one-dimensional) vector field. But Lie algebras of vector fields satisfy some non-trivial identities.
The embedding problem of a (differential) Lie algebra into its Wronskian enveloping algebra seems to be quite harder than the classical situation and related to Lie algebras of (one-dimensional) vector field. But Lie algebras of vector fields satisfy some non-trivial identities.

It might be useful to tackle this problem by dividing it into two parts: first the embedding problem of a Lie algebra into its Jacobi envelope, and secondly the embedding problem of a Jacobi algebra into its differential envelope.
Open problems

1. Is there an explicit description of the free Jacobi algebra on a set? of the differential envelope of a Jacobi algebra?
Open problems

1. Is there an explicit description of the free Jacobi algebra on a set? of the differential envelope of a Jacobi algebra?

2. Does the Wronskian envelope of a differential Lie algebra admit a structure of a (commutative) Hopf (differential) algebra?
Open problems

1. Is there an explicit description of the free Jacobi algebra on a set? of the differential envelope of a Jacobi algebra?

2. Does the Wronskian envelope of a differential Lie algebra admit a structure of a (commutative) Hopf (differential) algebra? The terminal map \((g, d) \rightarrow (0)\) lifts to a differential algebra morphism \(\epsilon: W(g) \rightarrow W(0) \cong R\), hence \(W(g)\) is an augmented (differential) algebra.
Open problems

1. Is there an explicit description of the free Jacobi algebra on a set? of the differential envelope of a Jacobi algebra?

2. Does the Wronskian envelope of a differential Lie algebra admit a structure of a (commutative) Hopf (differential) algebra? The terminal map \((g, d) \rightarrow (0)\) lifts to a differential algebra morphism \(\epsilon: \mathcal{W}(g) \rightarrow \mathcal{W}(0) \simeq R\), hence \(\mathcal{W}(g)\) is an augmented (differential) algebra. The diagonal \(\delta: g \rightarrow g \times g\) provides a differential algebra map \(\Delta: \mathcal{W}(g) \rightarrow \mathcal{W}(g \times g)\).
Open problems

1. Is there an explicit description of the free Jacobi algebra on a set? of the differential envelope of a Jacobi algebra?

2. Does the Wronskian envelope of a differential Lie algebra admit a structure of a (commutative) Hopf (differential) algebra? The terminal map \((\mathfrak{g}, d) \to (0)\) lifts to a differential algebra morphism \(\epsilon: \mathcal{W}(\mathfrak{g}) \to \mathcal{W}(0) \cong \mathbb{R}\), hence \(\mathcal{W}(\mathfrak{g})\) is an augmented (differential) algebra. The diagonal \(\delta: \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}\) provides a differential algebra map \(\Delta: \mathcal{W}(\mathfrak{g}) \to \mathcal{W}(\mathfrak{g} \times \mathfrak{g})\). Is \(\mathcal{W}\) a comonoidal functor from the cartesian monoidal category of differential Lie algebras to the monoidal category of commutative differential algebras under their tensor product?