Generic Base Algebras and Universal Comodule Algebras for some finite-dimensional Hopf algebras

U.N. Iyer, C. Kassel

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E. Aljadeff and C. Kassell associated to any finite-dimensional Hopf algebra $H$ an algebra $B_H$ of rational fractions, which is a finitely generated smooth domain of Krull dimension equal to the dimension of $H$. The algebra $B_H$ is called the *generic base algebra* associated to $H$.
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The generic base algebra is known for very few Hopf algebras: it has been described for finite group algebras in a work by Aljadeff, Haile, Natapov and for the four-dimensional Sweedler algebra by C. Kassel.
Definitions and motivation

- E. Aljadeff and C. Kassel associated to any finite-dimensional Hopf algebra $H$ an algebra $B_H$ of rational fractions, which is a finitely generated smooth domain of Krull dimension equal to the dimension of $H$. The algebra $B_H$ is called the generic base algebra associated to $H$.
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- In this talk, I will describe $B_H$ for other finite-dimensional Hopf algebras such as the Taft algebras, the Hopf algebras $E(n)$ and certain monomial Hopf algebras, all natural generalizations of the Sweedler algebra. This is joint work with C. Kassel.

A theory of polynomial identities for comodule algebras was also worked out by Aljadeff and Kassel. It leads naturally to a universal comodule algebra $U_H$, the analogue of the "relatively free algebra" in the classical theory of polynomial identities. The subalgebra of coinvariants $V_H$ of $U_H$ maps injectively into $B_H$. In the few known cases, the injection turns $B_H$ into a localization of $V_H$. We show that this also holds for the Hopf algebras considered here. Finally for the same Hopf algebras we also describe a suitable central localization of $U_H$ as a $B_H$-module.
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Definitions and motivation

- Fix a ground field $k$ of characteristic zero. All vector spaces, all algebras are defined over $k$; similarly, all linear maps are supposed to be $k$-linear. The symbol $\otimes$ denotes the tensor product over $k$. 
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- Denote the coproduct of a Hopf algebra by $\Delta$, its counit by $\varepsilon$, and its antipode by $S$. We also use a Heyneman-Sweedler-type notation
  \[
  \Delta(x) = x_1 \otimes x_2
  \]
  for the image under $\Delta$ of an element $x$ of a Hopf algebra $H$, and we write
  \[
  \Delta^{(2)}(x) = x_1 \otimes x_2 \otimes x_3
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  for its image under the iterated coproduct $\Delta^{(2)} = (\Delta \otimes \text{id}_H) \circ \Delta$. 

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- A (right) $H$-comodule algebra over a Hopf $k$-algebra $H$ is an associative unital $k$-algebra $A$ equipped with a right $H$-comodule structure whose (coassociative, counital) coaction
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The subalgebra $A^H$ of coinvariants of an $H$-comodule algebra $A$ is the subalgebra

$$A^H = \{ a \in A \mid \delta(a) = a \otimes 1 \}.$$
Example of an $H$-comodule algebra: Twisted comodule algebra

A two-cocycle $\alpha$ on a Hopf algebra $H$ is a bilinear form $\alpha : H \times H \to k$ satisfying the cocycle condition

$$\alpha(x_1, y_1) \alpha(x_2 y_2, z) = \alpha(y_1, z_1) \alpha(x, y_2 z_2)$$

for all $x, y, z \in H$. We always assume that $\alpha$ is invertible (with respect to the convolution product) and normalized, i.e., $\alpha(x, 1) = \alpha(1, x) = \varepsilon(x)$ for all $x \in H$. 
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• Let $u_H$ be a copy of the underlying vector space of $H$. Denote the identity map $u$ from $H$ to $u_H$ by $x \mapsto u_x$ ($x \in H$). The **twisted comodule algebra** $\alpha H$ is defined as the vector space $u_H$ equipped with the product given by

$$u_x u_y = \alpha(x_1, y_1) u_{x_2 y_2}$$

for all $x, y \in H$. This product is associative thanks to the cocycle condition. As $\alpha$ is normalized, the unit of $\alpha H$ is $u_1$. 
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The algebra $\alpha H$ is an $H$-comodule algebra with coaction $\delta: \alpha H \to \alpha H \otimes H$ given for all $x \in H$ by

$$\delta(u_x) = u_{x_1} \otimes x_2.$$ 

One can check that the subalgebra of coinvariants of $\alpha H$ coincides with $k u_1$. 
Another example of an $H$-comodule algebra: The tensor algebra

Take a copy $X_H$ of $H$; the identity map from $H$ to $X_H$ sends an element $x \in H$ to the symbol $X_x$. The map $x \mapsto X_x$ is linear and is determined by its values on a linear basis of $H$. Now consider the tensor algebra on $X_H$:

$$T(X_H) = \bigoplus_{i \geq 0} T^i(X_H),$$

where $T^i(X_H) = (X_H) \otimes^i$. There is a tautological $H$-comodule algebra structure on $T(X_H)$ with coaction $\delta : T(X_H) \to T(X_H) \otimes H$ given on each generator $X_x$ by

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Definition: $H$-identity. Given an $H$-comodule algebra $A$, we say that an element $P \in T(X_H)$ is an $H$-identity for $A$ if $\mu(P) = 0$ for all $H$-comodule algebra maps $\mu : T(X_H) \to A$. 

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Denote the set of all $H$-identities for $A$ by $\text{Id}_H(A)$. The set $I_H(A)$ is a two-sided ideal, right $H$-coideal of $T(X_H)$, and it is preserved by all comodule algebra endomorphisms of $T(X_H)$. 
The quotient algebra $\mathcal{U}_H(A) = T(X_H)/I_H(A)$ is an $H$-comodule algebra such that the canonical surjection $T(X_H) \to \mathcal{U}_H(A)$ is a comodule algebra map. By definition, all $H$-identities for $A$ vanish in $\mathcal{U}_H(A)$, which is the biggest quotient of $T(X_H)$ for which this happens. We call $\mathcal{U}_H(A)$ the \textit{universal $H$-comodule algebra} attached to the $H$-comodule algebra $A$ (in the classical literature on polynomial identities, $\mathcal{U}_H(A)$ is called the \textit{relatively free algebra}).
Definitions and motivation

- The quotient algebra \( U_H(A) = T(X_H)/I_H(A) \) is an \( H \)-comodule algebra such that the canonical surjection \( T(X_H) \rightarrow U_H(A) \) is a comodule algebra map. By definition, all \( H \)-identities for \( A \) vanish in \( U_H(A) \), which is the biggest quotient of \( T(X_H) \) for which this happens. We call \( U_H(A) \) the \textit{universal \( H \)-comodule algebra} attached to the \( H \)-comodule algebra \( A \) (in the classical literature on polynomial identities, \( U_H(A) \) is called the \textit{relatively free algebra}).

- \textbf{\( H \)-identities when \( A \) is a twisted comodule algebra:} Aljadeff and Kassel have shown that the \( H \)-identities for \( \alpha H \) can be detected by a comodule algebra map

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\mu_\alpha : T(X_H) \rightarrow S(t_H) \otimes \alpha H ,
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where the details are as follows:
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where the details are as follows:

Consider a copy $t_H$ of $H$, identifying each $x \in H$ linearly with the symbol $t_x \in t_H$. Define $S(t_H)$ to be the symmetric algebra on the vector space $t_H$.

The algebra $S(t_H) \otimes ^\alpha H$ is generated by the symbols $t_x u_y$ ($x, y \in H$) as a $k$-algebra (we drop the tensor product sign $\otimes$ between the $t$-symbols and the $u$-symbols).
It is a comodule algebra whose coaction is $S(t_H)$-linear and extends the coaction of $\alpha H$:

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The algebra map $\mu_\alpha : T(X_H) \rightarrow S(t_H) \otimes \alpha H$ is defined for all $x \in H$ by

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**Aljadeff-Kassel**: An element $P \in T(X_H)$ is an $H$-identity for $\alpha H$ if and only if $\mu_\alpha(P) = 0$. In other words, $I_H(\alpha H) = \text{Ker} \mu_\alpha$. 
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Let $\mathcal{U}_H^\alpha = \mathcal{U}_H(\alpha H)$ and $I_H^\alpha = I_H(\alpha H)$. It follows from the previous item that $\mu_\alpha$ induces an injection of comodule algebras

$$\mathcal{U}_H^\alpha = T(X_H)/I_H^\alpha \hookrightarrow S(t_H) \otimes \alpha H,$$

and by abuse of notation denote this map by $\mu_\alpha$. 
The algebra $\mathcal{V}_H^\alpha = (\mathcal{U}_H^\alpha)^H$ is the subalgebra of coinvariants.
Definitions and motivation

- The algebra $V^\alpha_H = (U^\alpha_H)^H$ is the subalgebra of coinvariants.
- **Localizing the symmetric algebra:** Aljadeff-Kassel showed that there is a unique linear map $x \mapsto t_x^{-1}$ from $H$ to the field of fractions $\text{Frac}\ S(t_H)$ of the symmetric algebra $S(t_H)$ such that for all $x \in H,$

$$\sum_{(x)} t_{x_1} t_{x_2}^{-1} = \sum_{(x)} t_{x_1}^{-1} t_{x_2} = \varepsilon(x) 1.$$  

When $x$ is a *group-like* element of $H$, i.e., such that $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$, then $t_x^{-1} = 1/t_x$. 
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When $x$ is a group-like element of $H$, i.e., such that $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$, then $t_x^{-1} = 1/t_x$.

Denote by $S(t_H)_\Theta$ the subalgebra of $\text{Frac} \ S(t_H)$ generated by all elements $t_x$ and $t_x^{-1}$ ($x \in H$).
When $H$ is a pointed Hopf algebra, $S(t_H)\Theta$ has a simple description as the following localization of $S(t_H)$:

$$S(t_H)\Theta = S(t_H)\left[\frac{1}{t_x}\right]_{x \in G(H)}.$$
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$$S(t_H)_\Theta = S(t_H) \left[ \frac{1}{t_x} \right]_{x \in G(H)}.$$

Note, the algebra $S(t_H)_\Theta$ carries a commutative Hopf algebra structure with coproduct $\Delta$, counit $\varepsilon$ and antipode $S$ given for all $x \in H$ by

$$\Delta(t_x) = t_{x_1} \otimes t_{x_2}, \quad \Delta(t_x^{-1}) = t_{x_2}^{-1} \otimes t_{x_1}^{-1},$$

$$\varepsilon(t_x) = \varepsilon(t_x^{-1}) = \varepsilon(x) \text{ and } S(t_x) = t_x^{-1}.$$

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$$\varepsilon(t_x) = \varepsilon(t_{x^{-1}}) = \varepsilon(x) \text{ and } S(t_x) = t_x^{-1}.$$ 

This Hopf algebra is Takeuchi’s free commutative Hopf algebra on the coalgebra underlying $H$; it satisfies the following universal property: for any coalgebra map $f : H \to H'$ into a **commutative Hopf algebra** $H'$, there is a unique Hopf algebra map $\tilde{f} : S(t_H)_{\Theta} \to H'$ extending $f$, i.e., such that $\tilde{f}(t_x) = f(x)$ for all $x \in H$. 

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The generic base algebra: To a pair \((H, \alpha)\) consisting of a Hopf algebra \(H\) and a normalized convolution invertible two-cocycle \(\alpha\), we attach a bilinear map \(\sigma_\alpha : H \times H \to S(t_H)\_\Theta\) with values in the previously defined algebra \(S(t_H)\_\Theta\).
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The map \(\sigma_{\alpha}\) is given for all \(x, y \in H\) by

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\sigma_{\alpha}(x, y) = t_{x_1} t_{y_1} \alpha(x_2, y_2) t_{x_3 y_3}^{-1}.
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The map $\sigma_\alpha$ is given for all $x, y \in H$ by

$$\sigma_\alpha(x, y) = t_{x_1} t_{y_1} \alpha(x_2, y_2) t_{x_3 y_3}^{-1}.$$ 

We call $\sigma_\alpha$ the generic cocycle associated to $\alpha$. The cocycle $\alpha$ being invertible, so is $\sigma_\alpha$, with inverse $\sigma_\alpha^{-1}$ given for all $x, y \in H$ by

$$\sigma_\alpha^{-1}(x, y) = t_{x_1 y_1} \alpha^{-1}(x_2, y_2) t_{x_3}^{-1} t_{y_3}^{-1}.$$
Definitions and motivation

Define the \textit{generic base algebra} $\mathcal{B}_H^\alpha$ attached to the pair $(H, \alpha)$ to be the subalgebra of $S(t_H)_{\Theta}$ generated by the values of the generic cocycle $\sigma_\alpha$ and of its inverse $\sigma_\alpha^{-1}$.

Since $\mathcal{B}_H^\alpha$ sits inside $S(t_H)_{\Theta}$, it is a domain.

Kassel-Masuoka: If $H$ is finite-dimensional, then

1. $\mathcal{B}_H^\alpha$ is a finitely generated smooth Noetherian domain of Krull dimension equal to $\dim_k H$; and
2. $S(t_H)_{\Theta}$ is a finitely generated projective $\mathcal{B}_H^\alpha$-module, from which it follows that $S(t_H)_{\Theta}$ is integral over $\mathcal{B}_H^\alpha$.

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It can be shown that the map $\mu_\alpha$ induces an embedding of the subalgebra of coinvariants $\mathcal{V}_H^\alpha$ of $\mathcal{U}_H^\alpha$ into $\mathcal{B}_H^\alpha$:

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\[ \mathcal{V}_H^\alpha \hookrightarrow \mathcal{B}_H^\alpha. \]

• Say that a two-cocycle $\alpha$ is nice if $\mathcal{B}_H^\alpha$ is a localization of $\mathcal{V}_H^\alpha$. 
It can be shown that the map $\mu_\alpha$ induces an embedding of the subalgebra of coinvariants $V_H^\alpha$ of $\mathcal{U}_H^{\alpha}$ into $B_H^\alpha$:

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The generic base algebra $B_H^\alpha$ when $H = kG$ is a group algebra was given by Aljadeff, Haile, Natapov.
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A presentation by generators and relations of $B_H^\alpha$ when $H$ is the four-dimensional Sweedler algebra was given by Aljadeff and Kassel. They also showed that any two-cocycle on a group algebra or on the Sweedler algebra is nice.
Definitions and motivation

- It can be shown that the map $\mu_\alpha$ induces an embedding of the subalgebra of coinvariants $V^\alpha_H$ of $U^\alpha_H$ into $B^\alpha_H$:

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- Kassel and Masuoka showed that any two-cocycle on a cocommutative Hopf algebra is nice.
Functoriality:
Consider pairs \((H, \alpha)\), where \(H\) is a Hopf algebra and \(\alpha\) is a normalized convolution invertible two-cocycle. We define a map of such pairs \((H, \alpha) \to (H', \alpha')\) to be a Hopf algebra map \(\varphi : H \to H'\) such that 
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• Define an algebra map $\varphi_T : T(X_H) \to T(X_{H'})$ by

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\varphi_T(X_x) = X_{\varphi(x)} \quad (x \in H).
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Definitions and motivation

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- Then, the map \(\varphi_T\) induces a comodule algebra map \(\varphi_U : U_H^\alpha \rightarrow U_{H'}^{\alpha'}\).
**Functoriality:**
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- Define an algebra map \(\varphi_S : S(t_H) \otimes \alpha H \to S(t_{H'}) \otimes \alpha' H'\) by

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- Then, the map \(\varphi_T\) induces a comodule algebra map \(\varphi_U : U_H^\alpha \to U_{H'}^{\alpha'}\).

- Moreover, if \(\varphi_S\) is injective, then so is \(\varphi_U\).
The algebra map $\varphi_S$ sends $B^\alpha_H$ to $B^\alpha_{H'}$. 

Trivial cocycle:
The trivial two-cocycle is given by $\alpha_0: (x, y) \mapsto \epsilon(x) \epsilon(y)$ ($x, y \in H$).

In this case it follows from (4) that $\alpha_H$ coincides as a $H$-comodule algebra with $H$ itself, the coaction being the coproduct, and that the linear isomorphism $u: H \to uH$ is a Hopf algebra map. This allows us to write $x$ instead of $u(x)$ ($x \in H$).

When $\alpha = \alpha_0$, we write $I_H$ for $I_{\alpha_H}$, $U_H$ for $U_{\alpha_H}$, $V_H$ for $V_{\alpha_H}$, and $B_H$ for $B_{\alpha_H}$. 

U.N. Iyer, C. Kassel

Generic Base Algebras and Universal Comodule Algebras for some finite-dimensional Hopf algebras
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When \( \alpha = \alpha_0 \), we write \( I_H \) for \( I_\alpha^H \), \( U_H \) for \( U_\alpha^H \), \( V_H \) for \( V_\alpha^H \), and \( B_H \) for \( B_\alpha^H \).
Note, $I_H$ is the kernel of the comodule algebra map

$$\mu_0 : T(X_H) \rightarrow S(t_H) \otimes H$$

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Also when $\alpha = \alpha_0$, we write $\sigma^{\pm 1}$ instead of $\sigma^{\pm 1}_{\alpha}$. In this case we have
$$\sigma(x, y) = t_{x_1} t_{y_1}^{-1} t_{x_2 y_2}^{-1} \quad \text{and} \quad \sigma^{-1}(x, y) = t_{x_1 y_1}^{-1} t_{x_2}^{-1} t_{y_2}^{-1}.$$
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Kassel-Masuoka showed that the study of the generic base algebra attached to a non-trivial cocycle can be reduced to the study of the generic base algebra attached to the trivial cocycle.
This works as follows: given a convolution invertible two-cocycle $\alpha$ on $H$, define the Hopf algebra $L = \alpha H \alpha^{-1}$ as the coalgebra $H$ with the product

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Of special interest are the so called lazy two-cocycles. In this case, $L = H$. 
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For example, on a cocommutative Hopf algebra all two-cocycles are lazy, so that $B_H^\alpha = B_H$ for any two-cocycle $\alpha$ on such a Hopf algebra.
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We concentrate on Hopf algebras equipped with the trivial cocycle.
Taft Algebras

**Taft algebras:** Fix an integer $n \geq 2$. We assume that the ground field $k$ contains a primitive $n$-th root of unity $q$. 

The Taft algebra $H = H_n$ has the following presentation as a $k$-algebra:

$$H = k\langle x, y \mid x^n = 1, yx = qyx, y^n = 0 \rangle.$$ 

The set $\{x^i y^j\}_{0 \leq i, j < n}$ is a basis of the vector space $H$, which therefore is of dimension $n^2$. 

The algebra $H$ is a Hopf algebra with coproduct $\Delta$ and counit $\varepsilon$ defined by

$$\Delta(x) = x \otimes x, \quad \Delta(y) = 1 \otimes y + y \otimes x, \quad \varepsilon(x) = 1, \quad \varepsilon(y) = 0.$$ 

When $n = 2$, the algebra $H$ is the four-dimensional Sweedler algebra.
**Taft algebras:** Fix an integer $n \geq 2$. We assume that the ground field $k$ contains a primitive $n$-th root of unity $q$.

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Taft Algebras

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**The generic base algebra:**

The algebra $S(t_H)$ can be identified with the polynomial algebra on the indeterminates $t_{x^i y^j} (0 \leq i, j < n)$. 

$S(t_H)_{\Theta}$ is a Hopf algebra. The coproduct on an element $t_{x^i y^j}$ is given by

$$\Delta(t_{x^i y^j}) = \sum_{r=0}^j \left[ j \atop r \right] t_{x^i y^r} \otimes t_{x^{i+r} y^j-r}.$$
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**The generic base algebra:**

The algebra $S(t_H)$ can be identified with the polynomial algebra on the indeterminates $t_{x^i y^j} \ (0 \leq i, j < n)$. The localization $S(t_H)_\Theta$ of $S(t_H)$ is obtained from $S(t_H)$ by inverting the elements $t_1, t_x, t_{x^2}, \ldots, t_{x^{n-1}}$ corresponding to the group-like elements of $H$:

$$S(t_H)_\Theta = S(t_H) \left[ \frac{1}{t_{x^i}} \right]_{0 \leq i < n}.$$
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$$\Delta(t_{x^iy^j}) = \sum_{r=0}^{j} \begin{bmatrix} j \\ r \end{bmatrix} t_{x^iry^r} \otimes t_{x^{i+r}y^{j-r}}.$$
Consider the set $\Gamma_0$ consisting of the following $n$ elements of $S(t_H)$: $t_x t_x^{n-1}$ and $t_x^i /(t_x)^i$ for $0 \leq i < n$, $i \neq 1$; these elements are invertible in $S(t_H)^\Theta$ and we denote by $\Gamma_0^{-1}$ the set of inverses of the elements of $\Gamma_0$. 

\[ \text{Let } \Gamma_1 \text{ be the set consisting of the elements } t_x t_y t_z, \text{ where } j \neq 0 \text{ and } i + j + k \equiv 0 \pmod{n}; \text{ the cardinality of } \Gamma_1 \text{ is } n(n-1). \]
Consider the set $\Gamma_0$ consisting of the following $n$ elements of $S(t_H)$: 
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Let $\Gamma_1$ be the set consisting of the elements $t_{x^i,y} t_{x^k}$, where $j \neq 0$ and $i + j + k \equiv 0 \pmod{n}$; the cardinality of $\Gamma_1$ is $n(n-1)$. 

[Note: Additional content from the Iyer-Kassel paper is not fully transcribed in the image provided.]
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**Iyer-Kassel:** Let $n \geq 3$.
(a) The $n^2$ elements of $\Gamma_0 \cup \Gamma_1$ are algebraically independent.
(b) The generic base algebra $B_H$ is given by

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$$B_H = k[\Gamma_0, \Gamma_0^{-1}, \Gamma_1].$$

When $n = 2$, the elements $t_1$, $t_x^2$, $t_x t_y$, $t_{xy}$ are algebraically independent and

$$B_H = k[(t_1)^{\pm 1}, (t_x^2)^{\pm 1}, t_x t_y, t_{xy}].$$
As a $B_H$-algebra, we have

$$S(t_H) \cong B_H[t]/\langle t^n - (t_x)^n \rangle.$$ 

That is, $S(t_H) \Theta$ is a finite étale (hence integral) extension of $B_H$ and it is a free $B_H$-module of rank $n$. 


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**The universal comodule algebra:**

The tensor algebra $T(X_H)$ is the free algebra on the indeterminates $X_{x_i y_j}$ ($0 \leq i, j < n$). We will use the same notation for the image of $X_{x_i y_j}$ in $U_H = T(X_H)/I_H$. 

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U.N. Iyer, C. Kassel

**Generic Base Algebras and Universal Comodule Algebras for some finite-dim...**
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Note that the comodule algebra map $\mu_0 : T(X_H) \to S(t_H) \otimes H$ defined by induces an embedding of $U_H$ into $S(t_H) \otimes H$ and
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- an element $P$ belongs to the subalgebra $\mathcal{V}_H$ of coinvariants of $U_H$ if and only if $\mu_0(P) \in S(t_H) \otimes 1$.
- Moreover, since the center of $H$ is one-dimensional, $\mathcal{V}_H$ coincides with the center of $U_H$. 
Now, \( \mu_0(X_1) = t_1 \), hence \( X_1 \in \mathcal{V}_H \), and

\[
\mu_0(X_{x^i}) = t_{x^i} x^i
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Let $\mathcal{V}'_H$ (resp. $\mathcal{U}'_H$) be the localization of $\mathcal{V}_H$ (resp. of $\mathcal{U}_H$) obtained by inverting the central elements $X_1, X_{x^i}$ for all $i = 1, \ldots, n - 1$. 
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• We have $\mathcal{V}'_H = \mathcal{B}_H$. That is, the trivial cocycle of a Taft algebra is nice.
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Moreover, There is an isomorphism of algebras

$$\mathcal{U}'_H \cong \mathcal{B}_H \langle \xi, \eta \mid \xi^n = X_{x^n}, \eta^n = 0, \eta \xi - q \xi \eta = 0 \rangle,$$

where, the isomorphism is given by

$$f(\xi) = X_x \quad \text{and} \quad f(\eta) = X_y - \frac{t_y}{t_x} X_x.$$
The Hopf algebras $E(n)$: Fix an integer $n \geq 1$. The algebra $H = E(n)$ is generated by $n + 1$ elements $x, y_1, \ldots, y_n$ subject to the relations

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The algebra $H$ is a Hopf algebra with coproduct $\Delta$ and counit $\varepsilon$ defined by

$$\Delta(x) = x \otimes x, \quad \Delta(y_i) = 1 \otimes y_i + y_i \otimes x,$$

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For this Hopf algebra, \( S(t_H) \) is the polynomial algebra on the indeterminates \( t_{y_I} \) and \( t_{xy_I} \), where \( I \) runs over all subsets of \( \{1, \ldots, n\} \).
- The localization $S(t_H)_\Theta$ of $S(t_H)$ is obtained from $S(t_H)$ by inverting $t_1$ and $t_x$:

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Set $\Gamma_1$ consists of the elements $t_{y_I}$, $t_xt_{xy_I}$, where $|I| \geq 2$, and of the elements $t_xt_{y_I}$, $t_{xy_I}$, where $|I|$ is odd. Observe that these elements are all of degree 0.
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The $2^{n+1}$ elements of $\Gamma_0 \cup \Gamma_1$ are algebraically independent and the generic base algebra $B_H$ is given by

$$B_H = k[\Gamma_0, \Gamma_0^{-1}, \Gamma_1].$$
As a $B_H$-algebra, we have

$$S(t_H) \cong B_H[t]/\left(t^2 - (t_x)^2\right).$$
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Let $\mathcal{V}'_H$ (resp. $\mathcal{U}'_H$) be the localization of $\mathcal{V}_H$ (resp. of $\mathcal{U}_H$) obtained by inverting the central elements $X_1$ and $X_x^2$. 

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Generic Base Algebras and Universal Comodule Algebras for some finite-dimensional Hopf algebras
We have $B_H = V'_H$. That is, the trivial cocycle on $E(n)$ is nice.
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Let $\mathcal{A}_H$ be the $\mathcal{B}_H$-algebra generated by $\xi, \eta_1, \ldots, \eta_n$ subject to the relations

\[
\xi^2 = X^2_x, \quad \eta_1^2 = \cdots = \eta_n^2 = 0, \quad \eta_i \xi + \xi \eta_i = 0, \quad \eta_i \eta_j + \eta_j \eta_i = 0
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for all $i, j = 1, \ldots, n$. There is an isomorphism of algebras $\mathcal{U}'_H \cong \mathcal{A}_H$. 
Fix an integer $n \geq 2$. Assume that the ground field $k$ contains a primitive $n$-th root of unity, which we denote by $q$. 

Monomial Hopf algebras of type $I$:
Consider a triple $(G, x, \chi)$, where $G$ is a finite group, $x$ a central element of $G$ of order $n \geq 2$, and $\chi$ a character $G \to k^\times$ such that $\chi^n = 1$ and $\chi(x) = q$.

To such a triple one associates a Hopf algebra $H$, which is defined as an algebra with generators the elements $g$ of $G$ and an additional generator $y$; the defining relations are those of the group algebra $kG$ as well as $y^n = 0$, $yg = \chi(g)y$ for all $g \in G$. 

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• The algebra $H$ has a Hopf algebra structure such that the natural inclusion $\iota : kG \rightarrow H$ is a Hopf algebra map and

$$\Delta(y) = 1 \otimes y + y \otimes x, \quad \varepsilon(y) = 0, \quad S(y) = -yx^{n-1}.$$
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Observe that $U_{kG}$ (resp. $B_{kG}$) splits off $U_H$ (resp. $B_H$). Similarly, passing to the coinvariants, $V_{kG}$ splits off $V_H$. 
• **The generic base algebra:** By definition, $S(t_H)$ is the polynomial algebra on the variables $t_{gy^i}$, where $g \in G$ and $0 \leq i < n$. The elements of $G$ are the only group-like elements.
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We have

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The coproduct on $S(t_H)_\Theta$ is determined by

$$\Delta(t_{gy^i}) = \sum_{r=0}^{i} \left[ \begin{array}{c} i \\ r \end{array} \right] t_{gy^r} \otimes t_{gy^i-r}.$$
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Consider the set $\Gamma$ of cardinality $(n - 1)|G|$ consisting of the fractions $t_{gy^i}/t_{gx^i}$ where $g \in G$ and $0 < i < n$. 
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The set $\Gamma$ is algebraically independent over $B_{kG}$ and

$$B_{H} = B_{kG}[\Gamma].$$
**The universal comodule algebra** There is a localization $\mathcal{V}'_{kG}$ of $\mathcal{V}_{kG}$ such that $\mathcal{B}_{kG} = \mathcal{V}'_{kG}$. We define a localization $\mathcal{V}'_H$ of $\mathcal{V}_H$ by

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We have $\mathcal{V}'_H = \mathcal{B}_H$ and there is an algebra isomorphism

$$\mathcal{U}'_H = \mathcal{U}'_{kG} \ast k[\eta] / (\eta^n = 0, \eta X_g - \chi(g) X_g \eta = 0 \mid g \in G).$$


