Why coalgebras?
Generalizations of results on Frobenius algebras, Hopf algebras and compact groups via co-representation theory, and applications

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**$K$-algebra $A$**

$m : A \otimes A \to A$ & $u : K \to A$

- $A \otimes A \otimes A \xrightarrow{m \otimes \text{Id}_A} A \otimes A$
- $A \otimes A \xrightarrow{m} A$

- $\text{Id}_A \otimes m$
- $m$

**$K$-coalgebra $C$**

$\Delta : C \to C \otimes C$ & $\varepsilon : C \to K$

- $C \otimes C \otimes C \xleftarrow{\Delta \otimes \text{Id}_C} C \otimes C$
- $C \otimes C \xleftarrow{\Delta} C$

- $\text{Id}_C \otimes \Delta$
- $\Delta$

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*Gen Frobenius Alg. and Integrals*
**A-module**: \( M \) with an \( A \)-action \( A \otimes M \to M, \ (a, m) \mapsto a \cdot m \)

**C-comodule**: \( M \) with a \( C \)-coaction \( \rho : M \to M \otimes C, \ \rho(m) = \sum_i m_i \otimes c_i \)
Modules and Comodules (actions and coactions)

**A-module**: $M$ with an $A$ - action $A \otimes M \rightarrow M$, $(a, m) \mapsto a \cdot m$

**C-comodule**: $M$ with a $C$ - coaction $\rho : M \rightarrow M \otimes C$, $\rho(m) = \sum_i m_i \otimes c_i$

and, of course, some compatibility conditions: associative and unital for modules, “coassociative and counital” for comodules

One defines morphisms of comodules, by duality with the definition of morphisms of modules.
Let \( \eta : A \to \text{End}(V) \) a finite dimensional representation, \( v_i \) a basis of \( V \). Then \( \eta(a) = (a_{ij}) \). Denote \( \eta_{ij}(a) = a_{ij} \) and \( \eta(ab) = \eta(a)\eta(b) \) reads

\[
\eta_{ij}(ab) = \sum_k \eta_{ik}(a)\eta_{kj}(b).
\]

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R\((A) = \{ f : A \to K \mid f(ab) = \sum_i g_i(a)h_i(b) \text{ for some } g_i, h_i : A \to K \}\) = \(A_0\)

We have \( m^* : A^* \to (A \otimes A)^* \supseteq A^* \otimes A^* \), and \( R(A) = (m^*)^{-1}(A^* \otimes A^*) \)

Closely related situation: \( G \) - a (topological) group and \( \eta : G \to \text{Gl}_n(V) \) a (continuous) representation over \( \mathbb{C} \).

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So for $f \in R(A)$, well determined $\sum g_i \otimes h_i \in A^* \otimes A^*$; by standard linear algebra, in fact $\sum g_i \otimes h_i \in R(A) \otimes R(A)$, giving a comultiplication of $R(A)$.

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**Coalgebra of representative functions** or **Finite dual coalgebra** $R(A)$
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**Coalgebra of representative functions or Finite dual coalgebra $R(A)$**

**Proposition**

$R(A)$ is spanned by all $\eta_{ij}$, $\eta : A \to \text{End}(V)$, $v_i$ basis; also $f \in R(A) \iff \ker(f)$ contains a two-sided ideal of finite codimension.
To any $\eta : A \rightarrow \text{End}(V)$ representation (f.d. left $A$-module) associate a right $R(A)$-comodule $V$

$$v_i \mapsto \sum_j v_j \otimes \eta_{ji}$$

Conversely, to a right $R(A)$-comodule $V$, $\rho : V \rightarrow V \otimes R(A)$, write $\rho(v_i) = \sum_j v_j \otimes f_{ji}$ associate the left $A$-action

$$a \cdot v_i = \sum_i f_{ji}(a) v_j$$

$f : V \rightarrow W$ morphism of $A$-modules iff $R(A)$-comodules.
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**Theorem**

The categories $f.d.A$-mod and comod-$R(A)$ are equivalent.
Coalgebras, comodules, rational modules

$C$-coalgebra $\Rightarrow C^*$ is an algebra with the **convolution product**:

$$(f * g)(c) = \sum_i f(a_i)g(b_i), \text{ where } \Delta(c) = \sum_i a_i \otimes b_i$$
Coalgebras, comodules, rational modules

$C$-coalgebra $\Rightarrow C^*$ is an algebra with the \textbf{convolution product}:

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comod-$C \hookrightarrow C^*$-mod: for $m \in (M, \rho : M \rightarrow M \otimes C)$, $f \in C^*$ define

$$f \ast m = \sum_i f(c_i)m_i \text{ where } \sum_i m_i \otimes c_i = \rho(m).$$

(note: above $= \text{the same for } A \rightarrow R(A)^*$ morphism of algebras)
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$C$-comodules are called rational $C^*$-modules. Also, for any $C^*$-module $M$, define $\text{Rat}(M) =$ the largest rational submodule of $M$. 
The Fundamental Theorem of Coalgebras

**Theorem**

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So, any rational module is the sum of its finite dimensional submodules, and a coalgebra is the sum of finite dimensional subcoalgebras.
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So, any rational module is the sum of its finite dimensional submodules, and a coalgebra is the sum of finite dimensional subcoalgebras.

So $C = \lim_{\to} C_i$, $C_i$-finite dimensional $\Rightarrow C^* = \lim_{\leftarrow} C_i^*$, a **profinite algebra**.

In close analogy to profinite groups:
Theorem

The following is equivalent for an algebra $A$:

• $A$ is profinite ($A = \lim_{\leftarrow} A_i$, $A_i$ f.d.)

• $A$ is pseudocompact, i.e. it is a Hausdorff and complete topological algebra with a basis of nbhds of 0, consisting of ideals of finite codimension.

• $A = C^*$, for some coalgebra $C$.

Moreover, in this situation, the category of right $C$-comodules is in duality with that of pseudocompact right $A$-modules.

$C$-coalgebra: $C = \bigoplus_{i} E(S_i)^n_i$ in mod-$C^*$; $A = C^* = \prod_{i} E(S_i)^* n_i$ in $C^*$-mod; $E(S_i)$-injective indecomposable with simple socle; $E(S_i)^*$ (principal) projective indecomposable & local.
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The development of the theory of infinite dimensional Frobenius algebras

An algebra \( A \) called Frobenius if \( A \cong A^\ast \) as left \( A \)-modules. The definition comes from an old problem raised by Frobenius and who's solution leads to this equivalent characterization. Such algebras generalize the classical case: \( A = KG \), \( G \) finite group.

- Maschke: \( K = \mathbb{C} \) (or \( \text{char}(K) \not | |G| \)), \( CG \) semisimple.
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Frobenius Algebras

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- $\varphi_r : A \longrightarrow \text{End}_K(A); \quad \varphi_r(a) = x \mapsto ax$ - morphism of $K$-algebras. $a \cdot e_i = \sum_{j=1}^{n} a_{ij}e_j$. Then

$A \ni a \mapsto \alpha(a) = (a_{ij})_{i,j} \in M_n(K)$ is a morphism of algebras.
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  e_i \cdot a &= \sum_{j} b_{ji}e_j; \text{ again } A \ni a \mapsto \beta(a) = (b_{ij})_{i,j} \in M_n(K) \text{ is a morphism of algebras.}
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- $\varphi_l : A \longrightarrow \text{End}_K(A), \varphi_l(a) = (x \mapsto xa)$ - antimorphism of algebras.

- $e_i \cdot a = \sum_j b_{ji}e_j$; again $A \ni a \mapsto \beta(a) = (b_{ij})_{i,j} \in M_n(K)$ is a morphism of algebras.

- Frobenius’ question: when are the two representations $\alpha, \beta$ equivalent:

  when $\exists S \in M_n(K)$ such that $\beta(a) = S^{-1}\alpha(a)S, \forall a \in A$ ?.
Co-Frobenius coalgebras

Definition

A coalgebra $C$ is called right (left) co-Frobenius if there is a monomorphism $C \hookrightarrow C^*$ of right (left) $C^*$-modules. $C$ is called co-Frobenius if it is both left and right co-Frobenius.
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A coalgebra $C$ is called right (left) quasi-co-Frobenius, or shortly right QcF coalgebra, if there is a monomorphism $C \hookrightarrow (C^*)^{(l)}$ of right (left) $C^*$-modules. $C$ is called QcF coalgebra if it is both left and right QcF coalgebra.
• A Frobenius algebra is finite dimensional.

• A "left" Frobenius (QF) algebra is also "right" Frobenius (QF).

• A left co-Frobenius (QcF) coalgebra is not necessarily right co-Frobenius (QcF).

• A Hopf algebra is left co-Frobenius iff it is right co-Frobenius.

• A C-finite dimensional coalgebra is co-Frobenius if and only if $C^*$ is Frobenius.

• Also C finite dimensional coalgebra is QcF iff $C^*$ is a QF ring, i.e. artinian and self-injective (and then also cogenerator) $\iff$ "injectives=projectives".

• C QcF $\iff$ C is a projective generator in comodules (or C-comod).

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- $C$ coalgebra is left QcF iff $C$ is projective as left $C^*$-module. In this case, $C$ is also a generator for rational $C$-comodules & $C^*$ is right self-injective!
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A coalgebra is left QcF iff it is projective as left dual-module. In this case, C is also a generator for rational C-comodules & its dual is right self-injective!

C is QcF iff C is a projective generator in comod-C (or C-comod).
The Weak Isomorphism

Definition

(i) Let $\mathcal{C}$ be a category having products. We say that $M, N \in \mathcal{C}$ are weakly $\pi$-isomorphic if and only if there are some sets $I, J$ such that $M^I \simeq N^J$; we write $M \overset{\pi}{\sim} N$.

(ii) Let $\mathcal{C}$ be a category having coproducts. We say that $M, N \in \mathcal{C}$ are weakly $\sigma$-isomorphic if and only if there are some sets $I, J$ such that $M(I) \simeq N(J)$; we write $M \overset{\sigma}{\sim} N$.
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(ii) Let $C$ be a category having coproducts. We say that $M, N \in C$ are weakly $\sigma$-isomorphic if and only if there are some sets $I, J$ such that $M^{(I)} \simeq N^{(J)}$; we write $M \overset{\sigma}{\sim} N$. 
Quasi-co-Frobenius Coalgebras: One Main Result

**Theorem**

Let $C$ be a coalgebra. Then the following assertions are equivalent.

(i) $C$ is a QcF coalgebra.

(ii) $C_{\sigma} \cong \text{Rat}(C^*)$ or $C_{\pi} \cong \text{Rat}(C^*)$ in $C_M$.

(iii) $C(N) \cong \text{Rat}(C^*)(N)$ or $C_{\prod N} \cong C_{\prod N} \text{Rat}(C^*)$.

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(iii) $C^{(\mathbb{N})} \overset{\sim}{\sim} (\text{Rat}(C^*))^{(\mathbb{N})}$ or $\prod_{\mathbb{N}} C \overset{\sim}{\sim} \prod_{\mathbb{N}} \text{Rat}(C^*)$
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Applications: Co-Frobenius coalgebras

Theorem

A coalgebra $C$ is co-Frobenius if and only if $C \cong \text{Rat}(C^* C^*)$ as left $C^*$-modules, if and only if $C \cong \text{Rat}(C_{C^*}^*)$ as right $C^*$-modules.

• $A$-finite dimensional is Frobenius $\iff A \cong A^*$. 
• $A$-profinite, $A = C^*$ then $C$ is co-Frobenius $\iff C \cong \text{Rat}(C^*)$. In this situation we have $A \cong A^\vee$ as left (or right) $A$-modules! Here $A^\vee = \text{topological completion of } \text{Hom}_{\text{cont}}(A, K)$.
• $A$-profinite, $A = C^*$ then $C$ is Quasi-co-Frobenius $\iff C_{\sigma, \pi} \cong \text{Rat}(C^*)$. In this situation, $A^\pi \cong A^\vee$. 

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- $A$-profinite, $A = C^*$ then $C$ is Quasi-co-Frobenius $\iff C \overset{\sigma,\pi}{\sim} \text{Rat}(C^*)$. In this situation, $A \overset{\pi}{\sim} A^\vee$!
Theorem

Let $C$ be a coalgebra. Then the following assertions are equivalent:

(i) $C$ is co-Frobenius.

(ii) The functors $\text{Hom}_{C^*}(-, C^*) : C\text{-comod} \to C^*\text{-mod}$ and $\text{Hom}_{K}(-, K) : C\text{-comod} \to C^*\text{-mod}$ are isomorphic in the category of functors from $C\text{-comod}$ to $C^*\text{-mod}$.

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(ii) The functors $\text{Hom}_{C^*}(-, C^*) : C\text{-comod} \to C^*\text{-mod}$ and $\text{Hom}(-, K) : C\text{-comod} \to C^*\text{-mod}$ are isomorphic in the category of functors from $C\text{-comod}$ to $C^*\text{-mod}$.

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Categorical characterizations

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Definition

$H$ - Hopf algebra: an algebra $(H, m, u)$ and a coalgebra $(H, \Delta, \varepsilon)$ + an antipode $S : H \to H$ s.t. $\Delta : H \to H \otimes H$ and $\varepsilon : H \to K$ are morphisms of algebras & $S$ is convolution inverse to $\text{Id}$. 

Example $G = \text{compact group}$, $H = \mathbb{R}(G)$ Hopf algebra, comultiplication as before, multiplication of complex functions, $S(f)(x) = \frac{1}{x} f(1)$. $\int = \text{integral of the left Haar measure}$ then $\int x \cdot f = \int f = u^* (\int)$.
Hopf Algebras

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**Example**

\(G=\text{compact group, } H = R(G)\) Hopf algebra, comultiplication as before, multiplication of complex functions, \(S(f) = (x \mapsto f(x^{-1}))\).

\(\int\) = integral of the left Haar measure then \(\int |_{R(G)} : R(G) \rightarrow \mathbb{C}\) has \(\int x \cdot f = \int f = u^*(x) \int f\) \((u : \mathbb{C} \rightarrow R(G), G \rightarrow R(G)^* \overset{u^*}{\rightarrow} \mathbb{C})\).
Theorem (Lin, Larson, Sweedler, Sullivan)

If $H$ is a Hopf algebra, then the following assertions are equivalent.

(i) $H$ is a right co-Frobenius coalgebra.

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(iv) $\text{Rat} \left( H^* \otimes H^* \right) \neq 0$.

(v) $\int l \neq 0$.

(vi) $\dim \int l = 1$.

(vii) The left-right symmetric version of the above.

As a consequence of the techniques developed: a new proof of the bijectivity of the antipode.
Fundamental Results on Hopf Algebras

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As a consequence of the techniques developed: a new proof of the bijectivity of the antipode.
Let $C$ - coalgebra; let $M$ - right $C$-comodule. Define 
\[ \int_{l,M} = \text{Hom}_{\text{comod}-C}(C, M). \]
For finite dimensional comodules:
\[ \int_{l,M} = \text{Hom}^C(C, M) = \text{Hom}_{C^*}(M^*, C^*). \]
Model: left integrals in a Hopf algebra, $\int_l = \text{Hom}(H, K)$ ($K$ right comodule as before by $K \rightarrow K \otimes H$, $1 \mapsto 1 \otimes 1^H$).

- Was considered before.
- In Hopf algebras, uniqueness of integrals reads $\dim(\int_l) \leq 1 = \dim(K)$; existence (in co-Frobenius Hopf algebras) $\dim(\int_l) \geq 1 = \dim(K)$.

Calling these “spaces integrals” has roots in...
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Calling these “spaces integrals” has roots in... **compact groups**
Compact Groups and vector valued “quantum”-invariant Integrals

$G$ - compact group.

**Note**

One could think of a measure which has the feature that translation of a set $U$ by $a$ has a certain effect on its measure $\mu(U)$ determined by $a$ itself (we could think that the measure of the translation of $U$ by $a$ depends on the measure of $U$ in a way that is ”quantified” by $a$).
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Example

$$d\mu_t(x) = e^{itx}dx \text{ on } G = (\mathbb{R}, +)$$

$$\int_{\mathbb{R}} f(x + a)d\mu_t(x) = \int_{\mathbb{R}} f(x + a)e^{itx}dx = \int_{\mathbb{R}} (f(x)e^{it(x-a)})dx = e^{-ita}\int_{\mathbb{R}} f(x)d\mu_t(x)$$
Compact Groups and vector valued “quantum”-invariant Integrals

For general $G$, one would need $\int a \cdot f = \eta(a) \int f$ for some $\eta(a) \in \mathbb{C}$.

More generally, we can consider vector valued integrals $\int : R(G) \to \mathbb{C}^n$, that is,

$$\int f d\mu = \left( \begin{array}{c} \int f d\mu_1 \\ \vdots \\ \int f d\mu_n \end{array} \right)$$

and the quantum invariance $\int a \cdot f d\mu = \eta(a) \cdot \int f d\mu$, where $\eta : G \to GL_n(\mathbb{C})$. 
Compact Groups and vector valued “quantum”-invariant Integrals

Note
Since \( \eta(xy) \int f = \int xy \cdot f = \eta(x) \int y \cdot f = \eta(x) \eta(y) \int f \), we can see that \( \eta \) must be a (continuous!) representation of \( G \).
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\( V = \mathbb{C}^n \) is a (left!) rep. of \( G \) with \( \eta : G \to \text{End}(V) \) iff \( V \) is a right \( R(G) \)-comodule. Moreover, a linear map \( \varphi : V \to W \) is a morphism of \( G \)-modules iff it is a morphism of \( R(G) \)-comodules.
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\[ \int(x \cdot f) = \eta(x) \int f = x \cdot \int(f) \], \( \Rightarrow \int \in \text{Hom}^{R(G)}(R(G), V) \) so \( \int \in \int_{l,R(G)} \).
Existence and Uniqueness of algebraic Integrals

In analogy to Hopf algebras and compact groups, we may think of the existence and uniqueness properties for integrals:

“Existence of integrals”: \( \dim(\int_{I,M}) \geq \dim(M) \) (Hopf algebras: 
\( \dim(\int_{I}) \geq 1 = \dim K, \int_{I} = \int_{I,K} \ldots \))

“Uniqueness of integrals”: \( \dim(\int_{I,M}) \leq \dim(M) \) (Hopf algebras: 
\( \dim(\int_{I}) \leq 1 = \dim K, \int_{I} = \int_{I,K} \ldots \))
co-Frobenius properties and integrals

**Proposition**

If $C$ is left $QcF$ then:

(i) $\int_{I,T} \neq 0$ for all rational simple left $C^*$-modules $T \Leftrightarrow C$ is right $QcF$

(ii) $\dim(\int_{I,T}) \geq \dim(T)$ for all rational simple left $C^*$-modules $T \Leftrightarrow C$ is right $QcF$. 
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Proposition

left co-Frobenius $\Rightarrow$ uniqueness of left integrals and existence of right integrals for all finite dimensional rational modules.
Some other Main Results

**Theorem**

A coalgebra $C$ is co-Frobenius (both on the left and on the right) if and only if $\dim(\int_l M) = \dim(M)$ for all finite dimensional right $C$-comodules $M$, equivalently, $\dim(\int_r N) = \dim(N)$ for all finite dimensional left $C$-comodules $N$. 

**Corollary**

• “Another Proof for the existence and uniqueness of integrals of Hopf algebras and the equivalent characterizations”.

• Further characterizations of co-Frobenius coalgebras!
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C categorical characterization

\( C \text{ coalgebra, } A = C^* \).

**Generalized (quasi)Frobenius**

\[
\begin{align*}
\text{f.d.Rat} & \rightarrow \text{mod} \ A \\
\text{Hom}_A(\cdot, A) & \subseteq \Hom_A(\cdot, K) \\
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**Quasi-co-Frobenius** = weak isomorphism

**co-Frobenius** = isomorphism
Other Results

- Examples showing that the results are the best possible;
- Examples showing that all the possible inclusions between the above classes of coalgebras and other important ones (& combinations of left & right of these) are strict. Also, left QcF $\Rightarrow$ left semiperfect, but also right semiperfect (new)!
- Other connections and applications to compact groups;
- For algebras, the cogenerator and the self-injective do not imply each other. For coalgebras, projective (left) implies generator (right); we prove the converse is not true, and give the precise conditions when it is.
Antipodes

$H$-dual quasi-Hopf algebra (co-quasi Hopf): $H$ coassociative coalgebra but not necessarily associative as an algebra. Same compatibility.

$\varphi \in (H \otimes H \otimes H)^*$ - reassociator, invertible with respect to the convolution algebra structure of $(H \otimes H \otimes H)^*$. For all $h, g, f, e \in H$:

\[
\begin{align*}
    h_1(g_1 f_1)\varphi(h_2, g_2, f_2) &= \varphi(h_1, g_1, f_1)(h_2 g_2)f_2 \\
    1h &= h1 = h \\
    \varphi(h_1, g_1, f_1 e_1)\varphi(h_2 g_2, f_2, e_2) &= \varphi(g_1, f_1, e_1)\varphi(h_1, g_2 f_2, e_2)\varphi(h_2, g_3, f_3) \\
    \varphi(h, 1, g) &= \varepsilon(h)\varepsilon(g)
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$\exists$ a coalgebra antimorphism $S$ of $H$ and elements $\alpha, \beta \in H^*$ such that for all $h \in H$:

\[
\begin{align*}
  S(h_1)\alpha(h_2)h_3 &= \alpha(h)1, & h_1 \beta(h_2)S(h_3) &= \beta(h)1 \\
  \varphi(h_1 \beta(h_2), S(h_3), \alpha(h_4)h_5) &= \varphi^{-1}(S(h_1), \alpha(h_2)h_3, \beta(h_4)S(h_5)) = \varepsilon(h).
\end{align*}
\]
0 \neq t \in \int; \ kt \subseteq Rat_{H^*H^*} = Rat_{H_{H^*}} \text{ is a two sided ideal } \Rightarrow \ kt \text{ also has a left comultiplication } t \mapsto a \otimes t. \text{ i.e. } t \cdot \alpha = \alpha(a)t, \ \forall \alpha \in H^*.

a - the distinguished grouplike of \ H.

For \ M \in \mathcal{M}^H, \ let \ ^aM \in ^H\mathcal{M} \ be (well) defined by

\[ M \ni m \mapsto m_{-1}^a \otimes m_0^a = aS(m_1) \otimes m_0 \in H \otimes M \]
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\textbf{Note}

The map } p : H \rightarrow \text{Rat}(H^*), \; p(x) = x \mapsto t \text{ is a bijective morphism of left } H\text{-comodules (right } H^*\text{-modules). In fact, we have an isomorphism of left } H\text{-comodules } H \otimes \int_r \rightarrow \text{Rat}(H^*), \; (x, t) \mapsto (x \mapsto t)
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\textbf{Proposition}

The map } p : a^H \to Rat(H^*), \ p(x) = x \mapsto t \text{ is a surjective morphism of left } H\text{-comodules (right } H^*\text{-modules).}
Theorem (Radford)
The antipode of a co-Frobenius Hopf algebra is bijective.

Proof. Only need $S$ surjective (the map $H\ni x \mapsto t \leftarrow x \in H^*$ is injective $\Rightarrow S$-injective)

Put $\pi := a_H \rightarrow \text{Rat}(H^*H^*) \sim \rightarrow H \otimes \int_r \approx H$; it splits ($H$ projective in $H\text{M}$) so $\exists \phi \in H\text{M}$ s.t. $\pi\phi = \text{Id}_H$.

$\phi(x) a^{-1} \otimes \phi(x) a_0 = x_1 \otimes \phi(x_2) \Rightarrow aS(\phi(x)_2) \otimes \phi(x)_1 = x_1 \otimes \phi(x_2) \Rightarrow S(a^{-1}) S(\phi(x)_2) \varepsilon\pi(\phi(x)_1) = x_1 \varepsilon\pi\phi(x_2) = x_1 \varepsilon(x) = x \Rightarrow x = S(\varepsilon\pi(\phi(x)_1)) \phi(x)_2 a^{-1}$.

□

This proof adapts to co-quasi Hopf algebras (dual quasi-Hopf algebras), with some technicalities; some assembly (inventivity) required...
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\]
Theorem (Radford)

The antipode of a co-Frobenius Hopf algebra is bijective.

Proof. [New] Only need $S$ surjective (the map $H \ni x \mapsto t \leftarrow x \in H^*$ is injective $\Rightarrow S$-injective)

Put $\pi := \overset{a}{H} \xrightarrow{p} \text{Rat}(H^*_H) \xrightarrow{\sim} H \otimes \int_r \simeq H$; it splits ($H$ projective in $H \mathcal{M}$) so $\exists \varphi \in H \mathcal{M}$ s.t. $\pi \varphi = \text{Id}_H$.

\[
\begin{align*}
\varphi(x)^a_1 \otimes \varphi(x)^a_0 &= x_1 \otimes \varphi(x_2) \Rightarrow \\
aS(\varphi(x)_2) \otimes \varphi(x)_1 &= x_1 \otimes \varphi(x_2) \Rightarrow \\
S(a^{-1})S(\varphi(x)_2)\varepsilon\pi(\varphi(x)_1) &= x_1\varepsilon\pi\varphi(x_2) = x_1\varepsilon(x_2) = x \Rightarrow \\
x &= S(\varepsilon\pi(\varphi(x)_1)\varphi(x)_2)a^{-1}).
\end{align*}
\]

This proof adapts to co-quasi Hopf algebras (dual quasi-Hopf algebras), with some technicalities; some assembly (inventivity) required...
A proof of the bijectivity of the antipode without the use of the uniqueness of integrals, which follows then as a consequence. This shows a much tighter connection to compact groups than realized before.

For $(M, \rho) \in \mathcal{M}^H$, $\rho : M \longrightarrow M \otimes H$, $\rho(m) = m_0 \otimes m_1$, we define $S_M \in \mathcal{H} \mathcal{M}$ with comodule structure given by

$$ m \mapsto m_{(-1)} \otimes m_{(0)} = S(m_1) \otimes m_0 $$
A proof of the bijectivity of the antipode without the use of the uniqueness of integrals, which follows then as a consequence. This shows a much tighter connection to compact groups than realized before.

For \((M, \rho) \in \mathcal{M}^H\), \(\rho : M \rightarrow M \otimes H\), \(\rho(m) = m_0 \otimes m_1\), we define \(S^M \in H^\mathcal{M}\) with comodule structure given by

\[ m \mapsto m_{(-1)} \otimes m_{(0)} = S(m_1) \otimes m_0 \]

**Proposition**

\(S^{Rat}(H^*)\), with left \(H\)-module structure given by

\[ H \otimes S^{Rat}(H^*) \rightarrow S^{Rat}(H^*), \quad x \otimes \alpha \rightarrow x \mapsto \alpha \]

and left \(H\)-comodule structure as above is a left \(H\)-Hopf module.
By the above and the Fundamental Th of Hopf modules:
\[ S\text{Rat}(H^*) \cong H \otimes (S\text{Rat}(H^*))^{co} = H \otimes \int_l \] and then we get a map
\[ \pi : (S\text{H})^{(\dim \int_l)} \cong S\text{Rat}(H^*) \cong H \otimes (S\text{Rat}(H^*))^{co} \to HH \]

Then, looking at the “coalgebras of the coefficients”, we get \( C_H \subseteq C_{S\text{H}} \) and then immediately \( H \subseteq S(H) \).

**New Perspective:** With this, the classical proof of the uniqueness of the Haar measure can be adopted “mutatis-mutandis” to Hopf algebras.


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