Differential Algebraic Geometry, Part I

Phyllis Joan Cassidy

City College of CUNY

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Abstract

- *Differential algebraic geometry*: A new geometry.
- *Founded on*: Commutative differential algebra (J. F. Ritt, 1930).
- *Differential algebraic varieties*: Solution sets of algebraic differential equations.
- *Model:* Algebraic geometry.
- *Geometric points for Ritt*: n-tuples of functions meromorphic in a region of $\mathbb{C}^m$.
- *Aim*: Unify and clarify the 19th century theory of algebraic differential equations.
- *Ritt's Focus*: Algorithms, similar to Buchberger's in Gröbner basis theory – designed to decide ideal membership; simplify differentiation-elimination.

- *Differential algebraic group theory:* Group objects in this new geometry.
- *Galois groups in a generalized differential Galois theory:* Fundamental matrices in Picard-Vessiot theory depend on parameters.
- *Central in Buium-Pillay-Hrushovski approach to Diophantine problems over function fields.*
- *Symmetry groups of systems of algebraic differential equations.*

- *Ellis R. Kolchin:* Ritt's geometry with a Weil approach.
- *Kolchin topology:* Adaptation of the Zariski topology.
- *Geometric points:* n-tuples with coordinates in a differential field.
- *Kolchin axiomatic treatment – abstract differential algebraic varieties* Emphasis –specializations of generic points.
- *Jerry Kovacic's differential schemes:* Framework– Grothendieck theory of schemes.

We begin: The Ritt-Kolchin theory of affine differential algebraic geometry. Time permitting: Kovacic's *differential schemes*.

Commutative differential algebra

All rings contain the field $\mathbb{Q}$ of rational numbers, and are associative, commutative, with unit 1. The 0 ring is the only ring for which $1 = 0$.

Definition

Let Θ be the free commutative monoid on the set $\Delta = \{\delta_1, \ldots, \delta_m\}$ of *derivation operators*. The elements of the monoid Θ are called *derivative operators*. The derivative operator

$$\theta = \delta_1^{i_1} \ldots \delta_m^{i_m}$$

has *order* $r = i_1 + \cdots + i_m$. Denote by $\Theta(r)$ the set of all $\theta \in \Theta$ whose order is $\leq r$.

Definitions

A ring $\mathcal{R}$ is a Δ-*ring* if there is a map from Δ into the multiplicative monoid End $(\mathcal{R}, +)$, with the additional conditions that for $\delta, \delta' \in \Delta$,

$$\delta\delta' = \delta'\delta,$$

and

$$\delta(ab) = a\delta b + b\delta a, \quad a, b \in \mathcal{R}, \quad \delta \in \Delta.$$

Definition

Δ-subrings and extension rings are defined in such a way that the actions of Δ are compatible. We refer to a Δ-extension ring of a Δ-ring $\mathcal{R}$ as a *Δ-R-algebra*.

Definition

The set $\mathcal{R}^\Delta$ of $c \in \mathcal{R}$ with $\delta c = 0\,, \quad \delta \in \Delta$, is a Δ-subring of $\mathcal{R}$ called the *ring of constants* of Δ. If $\mathcal{R}$ is a Δ-field, $\mathcal{R}^\Delta$ is a Δ-subfield.

The action of Δ on a Δ-ring $\mathcal{R}$ extends uniquely to a homomorphism from Θ into the multiplicative monoid End $(\mathcal{R}, +)$. This homomorphism maps Δ into Der $(\mathcal{R})$.

1. $1 \in \mathcal{R}^{\Delta}$. For, if $\delta \in \Delta$, $\delta(1) = \delta(1.1) = 1 \cdot \delta(1) + \delta(1) \cdot 1 = \delta(1) + \delta(1)$. Thus, $\delta(1) = 0$.
2. If $a \in \mathcal{R}$ is invertible, then $\forall \delta \in \Delta$

$$\begin{aligned} 0 &= \delta(1) = \delta\left(a \cdot a^{-1}\right) = a\delta\left(a^{-1}\right) + \delta(a)\, a^{-1}. \\ \delta\left(a^{-1}\right) &= -\frac{\delta(a)}{a^2}. \\ \delta\left(\frac{b}{a}\right) &= \delta\left(b \cdot \frac{1}{a}\right) = b\delta\left(\frac{1}{a}\right) + \delta(b)\,\frac{1}{a} = -\frac{b\delta(a)}{a^2} + \frac{\delta(b)}{a} \end{aligned}$$

So, we have the quotient rule

$$\delta\left(\frac{b}{a}\right) = \frac{a\delta(b) - b\delta(a)}{a^2}.$$

If a Δ-ring $\mathcal{R}$ is an integral domain, its Δ-ring structure extends uniquely to the quotient field of $\mathcal{R}$.

Definition

Let $z = (z_1, \ldots, z_n)$ be a family of elements of a Δ-$\mathcal{R}$-algebra. The Δ-$\mathcal{R}$-algebra

$$\mathcal{R}\{z\} = \mathcal{R}[\Theta z] = \varinjlim \mathcal{R}[\theta z]_{\text{ord } \theta \le r}.$$

It is said to be Δ-*finitely generated* by z. If $z_1, \ldots, z_n$ lie in a Δ-extension field of a Δ-field $\mathcal{F}$, the Δ-$\mathcal{F}$-extension

$$\mathcal{F}\langle z\rangle = \mathcal{F}(\Theta z) = \varinjlim \mathcal{F}(\theta z)_{\text{ord } \theta \le r}.$$

It is said to be Δ-*finitely generated* by z.

Example

Let $\mathcal{F} = \mathbb{C}(x, t)$, $\Delta = \{\partial_x, \partial_t\}$. Let

$$\mathcal{G} = \mathcal{F}\left\langle x^{t-1}e^{-x}\right\rangle,$$

where we have chosen a Δ- extension field of meromorphic functions, containing $x^{t-1}e^{-x}$.

$$x^{t-1}e^{-x} = e^{(t-1)\log x - x}$$

$$\begin{aligned} \partial_x\left(x^{t-1}e^{-x}\right) &= x^{t-1}e^{-x}\left(\frac{t-1-x}{x}\right). \\ \partial_t\left(x^{t-1}e^{-x}\right) &= x^{t-1}e^{-x}\log x. \end{aligned}$$

$$\mathcal{G} = \mathbb{C}(x, t)\left(x^{t-1}e^{-x}, \log x\right).$$

Let

$$\mathcal{H} = \mathcal{F}\left\langle \gamma \right\rangle, \quad \gamma = \int_0^x s^{t-1} e^{-s} ds,$$

where we have chosen an appropriate Δ-extension field of $\mathcal{F}$.

$$\partial_x \gamma = x^{t-1} e^{-x}.$$

$$\partial_t \gamma = \int_0^x (\log s) s^{t-1} e^{-s} ds.$$

$$\mathcal{H} = \mathbb{C}\left(x, t\right)\left(x^{t-1} e^{-x}, \log x\right)\left(\gamma, \partial_t \gamma, \partial_t^2 \gamma, ...\right).$$

The "special function" $\gamma = \gamma\left(x, t\right)$ is called the *(lower) incomplete gamma function*, and is prominent in statistics and physics. The family $\left(\gamma, \partial_t \gamma, \partial_t^2 \gamma, ...\right)$ is algebraically independent over $\mathcal{G}$ (Hölder 1887 (complete gamma), Johnson, Rubel, Reinhart 1995 incomplete gamma).

The differential polynomial algebra

Theorem

Let $\mathcal{R}$ be a Δ-ring. Let

$$(y_{i\theta})_{1\leq i\leq n,\theta\in\Theta}.$$

be a family of indeterminates over $\mathcal{R}$. There is a unique structure of Δ-ring on the polynomial ring $\mathcal{S}=\mathcal{R}\left[(y_{i\theta})_{1\leq i\leq n,\theta\in\Theta}\right]$ extending the Δ-ring structure on $\mathcal{R}$ and satisfying the condition that for every $\delta\in\Delta$, and pair (i,θ)

$$\delta y_{i\theta}=y_{i,\delta\theta}.$$

Note: By definition, $y_{i,\theta\theta'}=y_{i,\theta'\theta}$.

Example

$\mathcal{R} = \mathbb{Z}[x,t], \Delta = \{\partial_x \partial_t\}, n = 1.$ $\mathcal{S} = \mathbb{Z}[x,t][y, y_x, y_t, y_{xx}, y_{xt}, y_{tt}, ...].$
$P = xy^3 + xt^2 y y_x^3 y_t^{29}$. Set $\delta = \partial_x$. Extend δ to $\mathcal{S}$.
Want:

$$\begin{aligned} \partial_x y &= y_x, \\ \partial_x y_x &= y_{xx}, \\ \partial_x y_t &= y_{xt}. \end{aligned}$$

The proof will be broken up into lemmas.

Lemma

There is a unique derivation ∇ *on* $\mathcal{S}$ *such that* $\nabla\mid_R = \delta$*, and*

$$\nabla y_{i\theta} = 0$$

for every pair (i, θ)*.*

Proof.

Let $\delta \in \Delta$. For $P \in \mathcal{S}$, let P^δ be the polynomial obtained by differentiating the coefficients of P.
Let $\mathfrak{M}$ be the monomial basis of $\mathcal{S}$. Let

$$P = \sum_{M \in \mathfrak{M}} a_M M, \quad a_M \in \mathcal{R}, \quad \forall M \in \mathfrak{M}, \quad a_M = 0, \quad a\forall M.$$

$$\nabla P = P^\delta = \sum_{M \in \mathfrak{M}} (\delta a_M) M.$$

∇ is a derivation on $\mathcal{S}$ with the desired properties. □

Example

$\mathcal{R} = \mathbb{Z}[x, t], \Delta = \{\partial_x \partial_t\}, n = 1.$ $\mathcal{S} = \mathbb{Z}[x, t][y, y_x, y_t, y_{xx}, y_{xt}, y_{tt}, ...].$
$P = xy^3 + xt^2 y y_x^3 y_t^{29}.$ Set $\delta = \partial_x$
$\nabla P = y^3 + t^2 y y_x^3 y_t^{29}.$

Lemma

There is a unique derivation D on $\mathcal{S}$ such that $D\,|_{\mathcal{R}} = 0$ and

$$Dy_{i\theta} = y_{i,\delta\theta}.$$

Proof.

Define

$$\begin{aligned} DP &= \sum_{1\leq i\leq n,\theta\in\Theta} \frac{\partial P}{\partial y_{i\theta}} y_{i,\delta\theta} \\ Dy_{i\theta} &= y_{i,\delta\theta}. \end{aligned}$$

D is a derivation on $\mathcal{S}$ with the desired properties. □

Example

$\mathcal{R} = \mathbb{Z}[x, t], \Delta = \{\partial_x \partial_t\}, n = 1.$ $\mathcal{S} = \mathbb{Z}[x, t][y, y_x, y_t, y_{xx}, y_{xt}, y_{tt}, ...]$.
$P = xy^3 + xt^2 y y_x^3 y_t^{29}$. Set $\delta = \partial_x$

$$\begin{aligned} DP &= \frac{\partial P}{\partial y} y_x + \frac{\partial P}{\partial y_x} y_{xx} + \frac{\partial P}{\partial y_t} y_{xt} \\ &= 3xy^2 y_x + xt^2 y_x^3 y_t^{29} + 3xt^2 y y_x^2 y_t^{29} + 29xt^2 y y_x^3 y_t^{28}. \end{aligned}$$

Lemma

For $\delta \in \Delta$, define the extension of δ to $\mathcal{S} = \mathcal{R}\left[(y_{i\theta})_{1 \leq i \leq n, \theta \in \Theta}\right]$ to be the derivation

$$\delta = \nabla + D.$$

This definition extends the action of Δ from the coefficient ring to the polynomial algebra.

Proof.

By abuse of language, write

$$D = \sum_{1 \leq i \leq n, \theta \in \Theta} \frac{\partial}{\partial y_{i\theta}} y_{i,\delta\theta},$$

If

$$P = \sum_{M \in \mathfrak{M}} a_M M.$$

$$\delta P = \sum_{M \in \mathfrak{M}} (\delta a_M) M + \sum_{i,\theta} \frac{\partial P}{\partial y_{i\theta}} y_{i,\delta\theta}.$$

Let $\delta' = \nabla + D'$, where $D' = \sum_{M \in \mathfrak{M}} y_{i,\delta'\theta} \frac{\partial}{\partial y_{i\theta}}$.

$$\left[\delta, \delta'\right] \mid_R = 0.$$

$$\left[\delta, \delta'\right] (y_{i\theta}) = D(y_{i,\delta'\theta}) - D' (y_{i,\delta\theta}) = y_{i,\delta'\delta\theta} - y_{i,\delta\delta'\theta} = 0. \quad \left[\delta, \delta'\right] = 0.$$

□

Example: The Heat equation

$\Delta = \{\partial_x, \partial_t\}$

$$H = \partial_x^2 y - \partial_t y.$$

card $\Delta = 2$

Definition

Let $\mathcal{P} = \mathcal{R}\{y\}$ be the differential polynomial algebra. Let $F \in \mathcal{P}$. If $F \in \mathcal{R}$, we say the *order of* F is -1, If $F \notin \mathcal{R}$, then the *order of* F is the highest order derivative θy_j dividing a monomial of F.

The order of H is 2.

Definition

Let $\mathcal{R}$ be a Δ-ring. A family $z = (z_1, \ldots, z_n)$ of a Δ-$\mathcal{R}$-algebra is *Δ-algebraically dependent over $\mathcal{R}$* if the family Θz is algebraically dependent over $\mathcal{R}$.

The single element z is called Δ-algebraic over $\mathcal{R}$ if the family whose only element is z is Δ-algebraically dependent over $\mathcal{F}$.

Let $\Delta = \{\partial_x, \partial_t\}$. The incomplete gamma function

$$\gamma = \int_0^x s^{t-1} e^{-s} ds$$

is ∂_t-algebraically independent (∂_t-transcendentally transcendental) over both $\mathcal{F} = \mathbb{C}(x, t)$ and $\mathcal{G} = \mathbb{C}\left(x, t, x^{t-1} e^{-x}, \log x\right)$.
γ is ∂_x-algebraic over $\mathcal{F}$. It is a solution of the parametric linear homogeneous differential equation

$$\partial_x^2 y - \frac{t-1-x}{x} \partial_x y = 0,$$

Defining differential equations of the incomplete gamma function:

$$\begin{aligned}\partial_x^2 y - \frac{t-1-x}{x}\partial_x y &= 0,\\ \partial_x y \partial_t^2 \partial_x y - (\partial_t \partial_x y)^2 &= 0.\end{aligned}$$

Note that the family $(x^{t-1}e^{-x}, \log x)$ is algebraically independent over $\mathcal{F}$, but each of the elements is Δ-algebraically dependent over $\mathcal{F}$.

Differential ideals

1. What do we mean by "all differential consequences of a system

$$P_i = 0 \quad (i \in I)$$

of differential polynomial equations?" (Drach, Picard)

2. What do we mean by the defining differential equations of γ? (Drach, Picard)

Ritt's first answer to the first question: Consider the ideal in the differential polynomial ring generated by the P_i and all their derivatives.

Definition

An ideal $\mathfrak{a}$ of a Δ-ring $\mathcal{R}$ is a Δ-*ideal* if it is stable under Δ:

$$a \in \mathfrak{a} \Longrightarrow \delta \mathfrak{a} \in \mathfrak{a}, \quad \delta \in \Delta.$$

Definition

Let $\mathcal{R}$ be a Δ-ring.

1. $\mathfrak{I}(\mathcal{R})$ is the set of all Δ-ideals of $\mathcal{R}$.
2. $\mathfrak{R}(\mathcal{R})$ is the set of all radical Δ-ideals of $\mathcal{R}$.
3. $\mathfrak{P}(\mathcal{R})$ is the set of all prime Δ-ideals of $\mathcal{R}$.

$\mathfrak{P}(\mathcal{R}) \subset \mathfrak{R}(\mathcal{R}) \subset \mathfrak{I}(\mathcal{R})$
When we put a topology on $\mathfrak{P}(\mathcal{R})$, we will call it diffspec $(\mathcal{R})$.

Example

$\mathcal{R} = \mathbb{Q}[x], \quad \delta = \frac{d}{dx}$. Let $\mathfrak{p} \in \mathfrak{P}(\mathcal{R})$, $\mathfrak{p} \neq (0)$.

$$\mathfrak{p} = (P), P \text{ irreducible.}$$

Spose $\frac{dP}{dx} \neq 0$.

$$\deg \frac{dP}{dx} < \deg P, \text{ and } P \mid \frac{dP}{dx}.$$

Thus, $P \in \mathbb{Q} \rightarrowtail \leftarrowtail$. Therefore, $\operatorname{diffspec} \mathbb{Q}[x] = \operatorname{diffspec} \mathbb{Q}(x)$.

Some arithmetic of differential ideals

Let $\mathcal{R}$ be a Δ-ring.

Lemma

Let $(\mathfrak{a}_i)_{i \in I}$ be a family of elements of $\mathfrak{I}(\mathcal{R})$.

1. $\sum_{i \in I} \mathfrak{a}_i \in \mathfrak{I}(\mathcal{R})$.
2. $\bigcap_{i \in I} \mathfrak{a}_i \in \mathfrak{I}(\mathcal{R})$.
3. *If $\forall i \ \ \mathfrak{a}_i$ is radical, then, $\bigcap_{i \in I} \mathfrak{a}_i$ is radical.*

Lemma

Let $\mathcal{R}$ be a Δ-ring, and let $\mathfrak{a}$ and $\mathfrak{b}$ be in $\mathfrak{I}(\mathcal{R})$.

1. $\mathfrak{a}\mathfrak{b} \in \mathfrak{I}(\mathcal{R})$.
2. $\mathfrak{a} \cap \mathfrak{b} \in \mathfrak{I}(\mathcal{R})$, *and* $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$.

Homomorphisms of differential rings

Definition

Let $\mathcal{R}$ and $\mathcal{S}$ be Δ-rings. A homomorphism

$$\varphi : \mathcal{R} \longrightarrow \mathcal{S}$$

is a Δ-*homomorphism* if

$$\varphi \circ \delta = \delta \circ \phi, \quad \delta \in \Delta.$$

If $\mathcal{R}$ and $\mathcal{S}$ are Δ-$\mathcal{R}_0$-algebras, we call φ a Δ-$\mathcal{R}_0$-homomorphism if $\varphi \mid_{\mathcal{R}_0} = \mathrm{id}$.

Definition

Let $\mathcal{R}$ and $\mathcal{S}$ be Δ-rings, and let $\varphi : \mathcal{R} \longrightarrow S$ be a Δ-homomorphism.

1. ${}^{i}\varphi : \mathfrak{I}(\mathcal{S}) \longrightarrow \mathfrak{I}(\mathcal{R})$,
2. ${}^{r}\varphi : \mathfrak{R}(\mathcal{S}) \longrightarrow \mathfrak{R}(\mathcal{R})$,
3. ${}^{p}\varphi : \mathfrak{P}(\mathcal{S}) \longrightarrow \mathfrak{P}(\mathcal{R})$

are defined by the same formula $\mathfrak{b} \longmapsto \varphi^{-1}(\mathfrak{b})$.

Note that $\ker \varphi \in \mathfrak{I}(\mathcal{R})$. $\varphi(\mathcal{R})$ is a Δ-subring of $\mathcal{S}$.

Lemma

Let $\mathcal{R}$ and $\mathcal{S}$ be Δ-rings, and let $\varphi : \mathcal{R} \longrightarrow S$ be a surjective Δ-homomorphism.

1. $\ker \varphi \in \mathfrak{P}(\mathcal{R}) \Longleftrightarrow \mathcal{S}$ *is an integral domain.*
2. $\ker \varphi \in \mathfrak{R}(\mathcal{R}) \Longleftrightarrow \mathcal{S}$ *is reduced (no nonzero nilpotent elements).*
3. ${}^{i}\varphi$ *maps* $\mathfrak{I}(\mathcal{S})$ *bijectively onto the set of Δ-ideals of* $\mathcal{R}$ *containing* $\ker \varphi$.
4. ${}^{r}\varphi$ *maps* $\mathfrak{R}(\mathcal{S})$ *bijectively onto the set of radical Δ-ideals of* $\mathcal{R}$ *containing* $\ker \varphi$.
5. ${}^{p}\varphi$ *maps* $\mathfrak{P}(\mathcal{S})$ *bijectively onto the set of prime Δ-ideals of* $\mathcal{R}$ *containing* $\ker \varphi$.

In the last three statements, the maps are inclusion preserving and their inverses send $\mathfrak{a}$ to $\varphi(\mathfrak{a})$.

Lemma

Let $\mathfrak{a}$ be Δ-ideal in a Δ-ring $\mathcal{R}$. Then $\mathcal{R}/\mathfrak{a}$ has a unique structure of Δ-ring such that the quotient homomorphism $\pi : \mathcal{R} \longrightarrow \mathcal{S}$ is a Δ-homomorphism.

Proof.

For $\delta \in \Delta$ and $x \in \mathcal{R}$, set $\overline{x} = x + \mathfrak{a}$, and define $\delta\overline{x} = \overline{\delta x}$. Let $y \in \mathcal{R}$. Spose $\overline{x} = \overline{y}$.

$$\begin{aligned} x - y &\in \mathfrak{a}. \\ \delta(x-y) &= \delta x - \delta y \in \mathfrak{a}. \\ \delta\overline{x} &= \delta\overline{y}. \end{aligned}$$

So, the action of Δ on $\mathcal{R}/\mathfrak{a}$ is well-defined. The sum and product rules follow easily. □

Corollary

Let $\mathfrak{a}$ *be* Δ*-ideal in a* Δ*-ring* $\mathcal{R}$*. Let* π *be the quotient homomorphism.*

1. ${}^{i}\pi$ *maps* $\mathfrak{I}(\mathcal{R}/\mathfrak{a})$ *bijectively onto the set of* Δ*-ideals of* $\mathcal{R}$ *containing* $\mathfrak{a}$*.*
2. ${}^{i}\pi$ *maps* $\mathfrak{R}(\mathcal{R}/\mathfrak{a})$ *bijectively onto the set of radical* Δ*-ideals of* $\mathcal{R}$ *containing* $\mathfrak{a}$*.*
3. ${}^{i}\pi$ *maps* $\mathfrak{P}(\mathcal{R}/\mathfrak{a})$ *bijectively onto the set of prime* Δ*-ideals of* $\mathcal{R}$ *containing* $\mathfrak{a}$*.*

Differential ideal bases

Definition

Let $\mathcal{R}$ be a Δ-ring and $\mathfrak{a}$ be a Δ-ideal of $\mathcal{R}$. The *Δ-ideal* $\mathfrak{a}$ is *generated by a subset S* if the *ideal* $\mathfrak{a}$ is generated by ΘS.

We denote it by $[S]$. Call $\mathcal{S}$ a (Δ-ideal) basis of $\mathfrak{a}$. $[S]$ is the smallest Δ-ideal containing S.
Question (Drach, Picard): Is every system of differential polynomial equations equivalent to a finite system?
If $\mathcal{R}$ is a ring finitely generated over a field, every ideal of $\mathcal{R}$ is finitely generated. So, the answer is yes for polynomial equations.

Example

Let $\mathcal{R} = \mathcal{F}\{y\}$, $\mathcal{F}$ a Δ-field, $\Delta = \{\delta\}$, y a Δ-indeterminate over $\mathcal{F}$.
Write $y', y'', \ldots, y^{(i)}, \ldots$.

$$\mathfrak{i} = \left[yy', y'y'', \ldots, y^{(i)}y^{(i+1)}, \ldots \right]$$

has no finite Δ-ideal basis (Ritt, 1930 Also, see Kovacic-Churchill, Notes KSDA).

Radicals redux

Let $\mathcal{R}$ be a Δ-ring. Let $\mathfrak{a}$ be a Δ-ideal of $\mathcal{R}$.The intersection of the family of radical Δ-ideals of $\mathcal{R}$ containing $\mathfrak{a}$ is a radical Δ-ideal.
So, there is a *smallest radical Δ-ideal of $\mathcal{R}$ containing $\mathfrak{a}$.*
The radical $\sqrt{\mathfrak{a}}$ is the set of $a \in \mathcal{R}$ such that there is a positive integer n with $a^n \in \mathfrak{a}$. It is an ideal of $\mathcal{R}$, and is the smallest radical ideal of $\mathcal{R}$ containing $\mathfrak{a}$. Is it a Δ-ideal? Conjecture: Yes.

Example

Let $\mathcal{R} = \mathbb{Z}[x]$, $\delta = \frac{d}{dx}$.
Let $\mathfrak{a} = (2, x^2)$. $\mathfrak{a}$ is a δ-ideal of $\mathcal{R}$. Let $\mathcal{S} = \mathcal{R}/\mathfrak{a}$. $\overline{x} \in \sqrt{[0]}$. $\delta\overline{x} = 1 \notin \sqrt{[0]}$.

Is this a counterexample to the conjecture? No. Our Δ-rings are Ritt algebras.

Theorem

Let $\mathcal{R}$ be a Δ-ring (Ritt algebra), and let $\mathfrak{a}$ be a Δ-ideal of $\mathcal{R}$. Then, the radical of $\mathfrak{a}$ is a Δ-ideal of $\mathcal{R}$.

If $\mathfrak{a} = [S]$, call $\mathfrak{r} = \sqrt{\mathfrak{a}}$ the *radical Δ-ideal generated by S*. S is also called a *(radical Δ-ideal) basis* for the radical Δ-ideal $\mathfrak{r}$.

Proof.

Let $a \in \sqrt{\mathfrak{a}}$. Let $n \in \mathbb{Z}_{>0}$ be such that $a^n \in \mathfrak{a}$. Claim: For any $\delta \in \Delta$, $k = 0, \ldots, n$,

$$a^{n-k} (\delta a)^{2k} \in \mathfrak{a}.$$

By hypothesis, the case $k = 0$ is true. Let $0 \leq k \leq n - 1$. Assume true for k. Differentiate.

$$(n-k)\, a^{n-k-1} (\delta a)^{2k+1} + 2k a^{n-k} (\delta a)^{2k-1} \left(\delta^2 a\right) \in \mathfrak{a}$$

by the induction hypothesis.

$$\delta a[(n-k)\, a^{n-k-1} (\delta a)^{2k+1} + 2k a^{n-k} (\delta a)^{2k-1} \left(\delta^2 a\right)] \in \mathfrak{a}$$

$$a^{n-k-1} (\delta a)^{2k+2} \in \mathfrak{a}.$$

by the induction hypothesis, and, since $\mathcal{R}$ is a Ritt algebra. So, the claim is true for $k + 1$. Set $k = n$. □

The Ritt basis theorem

Theorem

Let $\mathcal{F}$ be a Δ-field, and $\mathcal{R} = \mathcal{F}\{z_1, \ldots, z_n\}$ be a finitely Δ-generated Δ-$\mathcal{F}$-algebra. Then, every radical Δ-ideal has a finite (radical Δ-ideal) basis.

Set $\mathcal{R} = \mathcal{F}\{y_1, \ldots, y_n\}, y_1, \ldots, y_n$ Δ-indeterminates. Let Σ be any subset of $\mathcal{R}$. The radical Δ-ideal $\mathfrak{r} = \surd\,[\Sigma]$ has a finite basis. There is a finite subset $F_1, \ldots, F_r$ of $\mathfrak{r}$ such that $\mathfrak{r} = \surd\,[F_1, \ldots, F_r]$. The radical Δ-ideal $\mathfrak{r} = \surd\,[\Sigma]$ is Ritt's final interpretation of "all differential consequences of the system

$$F = 0, \quad F \in \Sigma."$$

The basis theorem is his answer to Drach-Picard: Is every system of differential polynomial equations equivalent to a finite system?
The solution space of the system defined by Σ is also defined by

$$F_1 = 0. \ldots, F_r = 0.$$

Zeros of differential polynomials and ideals

Let $\mathcal{R}$ be a Δ-ring and $y = (y_1, \ldots, y_n)$ be a family of Δ-indeterminates over $\mathcal{R}$. Let $\mathcal{S} = \mathcal{R}\{y\}$.

$$\mathcal{S}_r = \mathcal{R}\left[\theta y\right]_{\theta \in \Theta(r)}.$$

Let $z = (z_1, \ldots, z_n) \in \mathcal{R}^n$. Then, $z \leftrightarrow (z, \delta_1 z, \ldots, \delta_m z, \ldots, \theta z, ...)$. On each polynomial ring $\mathcal{S}_r$ we have the substitution homomorphism

$$\mathcal{S}_r \longrightarrow \mathcal{R}, \quad (\theta y) \longmapsto (\theta z), \theta \in \Theta.$$

This defines a Δ-$\mathcal{R}$-homomorphism σ_z from $\mathcal{S}$ into $\mathcal{R}$, called the Δ*-substitution homomorphism*. For $P \in \mathcal{S}$, write $P(z)$ for $\sigma(P)(z)$, and call it the *value* of P at z. $\ker \sigma_z$ is a Δ-ideal of $\mathcal{S}$, called the *defining* Δ*-ideal* of z.

Example

Let $\Delta = \{\partial_x, \partial_t\}$, $\mathcal{F}$ the Δ-field of functions meromorphic in $\mathbb{D}_x \times \mathbb{D}_t$, where $\mathbb{D}_x$ is the right half plane of $\mathbb{C}$, $\mathbb{D}_t = \mathbb{C} \backslash \mathbb{Z}_{\leq 0}$. Let $\gamma = \int_0^x s^{t-1} e^{-s} ds \in \mathcal{F}$. The defining Δ-ideal of γ in $\mathcal{F}\{y\}$ is the prime Δ-ideal

$$\mathfrak{p} = \left[\partial_x^2 y - \frac{t-1-x}{x} \partial_x y, \partial_x y \partial_t^2 \partial_x y - (\partial_t \partial_x y)^2\right].$$

Definition

Let $\mathcal{R}$ be a Δ-ring, $y = (y_1, \ldots, y_n)$ a family of Δ-indeterminates over $\mathcal{R}$, $\mathcal{S} = \mathcal{R}\{y\}$. Let Σ be a subset of $\mathcal{S}$. The *zero set* of Σ is the set

$$Z = \{z \in \mathcal{R}^n : P(z) = 0, \quad P \in \Sigma\}.$$

Example

Set

$$\Sigma = \left\{ \partial_x^2 y - \frac{t-1-x}{x} \partial_x y, \partial_x y \partial_t^2 \partial_x y - (\partial_t \partial_x y)^2 \right\},$$

$\mathcal{F}$ as above. . Determine $Z \subset \mathcal{F}$. The zero set of $L = \partial_x^2 y - \frac{t-1-x}{x} \partial_x y$ is

$$\begin{aligned} V &= \{ c_0(t) + c_1(t)\gamma \}, \\ \gamma &= \int_0^x s^{t-1} e^{-s} ds \in \mathcal{F} \end{aligned}$$

Let

$$z = c_0(t) + c_1(t)\gamma.$$

Then, $\partial_x z = 0$ if and only if $c_1(t) = 0$. So, spose $c_1(t) \neq 0$. Then, z is a zero of the second polynomial

$$\partial_x y \partial_t^2 \partial_x y - (\partial_t \partial_x y)^2$$

if and only if

$$\partial_t (\ell \partial_t \partial_x z) = 0, \quad \ell \partial_t \partial_x z = \frac{\partial_t \partial_x z}{\partial_x z}.$$

$$\partial_x z = c_1(t)\partial_x \gamma = c_1(t) x^{t-1} e^{-x}.$$

$$\begin{aligned} \ell \partial_t \partial_x z &= \ell \partial_t c_1(t) + \ell \partial_t (x^{t-1} e^{-x}) \\ &= \ell \partial_t c_1(t) + \log x \end{aligned}$$

$$\partial_t (\ell \partial_t \partial_x z) = \partial_t \ell \partial_t c_1(t).$$

$$Z = \mathcal{F}^{\partial_x} \cdot 1 \cup (\mathcal{F}^{\partial_x} \cdot 1 + G \cdot \gamma),$$

where G is the subgroup of the multiplicative group of $\mathcal{F}^{\partial_x}$ satisfying the differential equation

$$\partial_t \left(\frac{\partial_t y}{y} \right) = 0.$$

$$G\gamma = k_1 e^{k_2 t} \int_0^x s^{t-1} e^{-s} ds, \quad k_1, k_2 \in \mathbb{C}.$$

G is a *differential algebraic group.*