

Solving linear differential equations

Marius van der Put and Nguyen Khuong An

(1) *Introduction.* L. Fuchs posed the problem: Consider the differential operator

$L = \partial^n + a_{n-1}\partial^{n-1} + \dots + a_1\partial + a_0$ with $a_i \in K = \mathbb{C}(z)$ and $\partial = \frac{d}{dz}$. Suppose that the n independent solutions y_1, \dots, y_n of the scalar differential equation $L(y) = 0$ satisfy a non trivial homogeneous equation $F(y_1, \dots, y_n) = 0$ over \mathbb{C} . Can one express the y_i in terms of solutions of scalar linear differential equations of lower order?

G. Fano wrote a long paper (1900) on this theme. His tools were an early form of the differential Galois group and an extensive knowledge of low dimensional projective varieties.

One of Fano's examples is (simplified):

A basis y_1, \dots, y_5 of the solutions of $L_5 = \partial^5 + 2p\partial^3 + 3p'\partial^2 + (3p'' + p^2 - 4q)\partial + (p''' + pp' - 2q')$ satisfies $\sum y_i^2 = 0$. All solutions for L_5 can be obtained from $L_4 = \partial^4 + \partial p\partial + q$. Indeed, any two solutions u_1, u_2 of L_4 yield the solution $u_1u_2' - u_1'u_2$ of L_5 .

In papers (1985-1988) of M.F. Singer and in a recent paper of K.A. Nguyen, a powerful combination of Tannakian methods and representations of semi-simple Lie algebras yields a complete answer to Fuchs' question.

We will explain this and then come to the theme of this talk:

Explicit methods for solving a differential equation in terms of equations of lower order, whenever possible.

(2) *Some differential Galois theory*

A scalar equation of order n can be transformed into a *matrix differential equation* $Y' = AY$ with $A \in \text{Matr}(n, K)$ and $Y \in K^n$.

It can be transformed further into a *differential module* $M = (M, \partial)$ over K , where M is an n -dimensional vector space over K and the additive map $\partial : M \rightarrow M$ satisfies $\partial(fm) = f'm + f\partial m$.

The differential module (M, ∂) obtained from the matrix equation $Y' = AY$ is $M = K^n$ and $\partial = \frac{d}{dz} - A$. The equation $\partial m = 0$ is the translation of $Y' = AY$.

Let M be a differential module over K of dimension n , corresponding to $Y' = AY$. There exists an extension of differential fields $K \subset L$, called the *Picard-Vessiot extension*, such that

- (i) There exists a $F \in GL(n, L)$ with $F' = AF$,
- (ii) L is generated over K by the entries of F ,
- (iii) L and K have the same field of constants (here \mathbb{C}).

The translation of (ii) in terms of M is:

$V = V(M) := \ker(\partial, L \otimes_K M)$ is a vector space over \mathbb{C} of dimension n . This is called the *solution space*.

The *differential Galois group* $G = Gal(M)$ of M is the group of the differential automorphisms of L/K . The induced action of G on $L \otimes_K M$ commutes with ∂ and thus G acts on $V(M)$. This action is faithful and embeds G as a linear algebraic group in $GL(V)$. Further $(\mathfrak{gal}(M), V)$ will denote the Lie algebra of $Gal(M)$ acting on the solution space V .

Example: $y^{(2)} = zy$ over $K = \mathbb{C}(z)$ has module form: $M = Ke_1 + Ke_2$ by $\partial e_1 = e_2$, $\partial e_2 = ze_1$. $L = K(y_1, y_1', y_2, y_2')$ with $y_i^{(2)} = zy_i$ for $i = 1, 2$ and with the only relation $y_1 y_2' - y_1' y_2 = 1$, is the Picard-Vessiot extension. $V = V(M)$ has basis $\{-y_i' e_1 + y_i e_2 \mid i = 1, 2\}$ over \mathbb{C} . Further $\text{Gal}(M) = \text{SL}(V)$ and $(\text{gal}(M), V) = (\mathfrak{sl}(V), V)$.

Definition: Let $d \geq 2$ be an integer. A differential module M is called d -solvable if there exist differential fields $K = K_1 \subset K_2 \subset \cdots \subset K_r$, with $[K_{i+1} : K_i] < \infty$ or is the Picard-Vessiot extension of a differential module of dimension $\leq d$, and K_r contains a Picard-Vessiot field for M .

Fuchs' question reads now: Suppose that a basis of the solutions y_1, \dots, y_n of an diff. eqn. $L(y) = 0$ satisfies some non trivial homogeneous equation $F(y_1, \dots, y_n) = 0$ over \mathbb{C} . Is L then $(n - 1)$ -solvable?

A (neutral) *Tannaka category* T is an abelian category having tensor products, internal homs etc. *Main result:* T is a Tannaka category if and only if T is equivalent to the category $Repr_G$ of the finite dimensional representations (here over \mathbb{C}) of an affine group scheme G .

The category Diff_K of all diff. mod. over K is a Tannaka category. For a fixed differential module M one considers the subcategory $\{\{M\}\}$ of Diff_K , generated by M and closed under all operations of linear algebra. Now $\{\{M\}\}$ is again a Tannaka category and equivalent to $Repr_G$ with $G = Gal(M)$. This equivalence can be made explicit by using the Picard-Vessiot field L , namely

$N \in \{\{M\}\} \mapsto \ker(\partial, L \otimes_K N) = V(N)$ and the latter is a $Gal(M)$ -representation.

(3) *Answer to Fuchs' question.*

Criterion. (M.F. Singer)

*Let $\dim_K M = n \geq 3$ and $Gal(M) \subset SL(V(M))$
(can be assumed without loss of generality).*

*Then M is not $(n-1)$ -solvable iff $(\mathfrak{gal}(M), V(M))$
satisfies: $\mathfrak{gal}(M)$ is a simple Lie algebra and
 $V(M)$ is a representation of smallest dimension.*

Sketch. We may suppose that $Gal(M)$ is connected (for this a finite extension of K is needed). Necessary is: M is (absolutely) irreducible.

Then $Gal(M)$ is a semi-simple algebraic group with semi-simple Lie algebra \mathfrak{g} . Now we need the following *non constructive* result.

Proposition. Let $G^+ \rightarrow G$ be a surjective morphism of connected linear algebraic groups over \mathbb{C} having a finite kernel Z . Let M be a differential module over K with $Gal(M) = G$. Suppose that K is a C_1 -field (if not, a finite extension of K is needed). Then there exists a differential module N over K with $Gal(N) = G^+$, such that the representation $(Gal(N), V(N))$ has minimal dimension and $M \in \{\{N\}\}$.

We apply this with $G = Gal(M)$ and G^+ is the simply connected group with Lie algebra \mathfrak{g} . Thus we may replace M by N or assume that $Gal(M)$ is simply connected.

Now we have equivalence of the categories $\{\{M\}\} \rightarrow Repr_{Gal(M)} \rightarrow Repr_{\mathfrak{gal}(M)}$. Hence the representation $V(M)$ of the semi-simple Lie algebra $\mathfrak{gal}(M)$ cannot be obtained by representations of smaller degree.

This implies that $\text{gal}(M)$ is semi-simple and that $V(M)$ is a representation of smallest dimension.

Fuchs' question has a negative answer if the corresponding representation $(\text{Gal}(M), V(M))$ admits a non trivial invariant homogeneous form F . The following list gives the complete answer.

Simple Lie algebras, smallest dimension, degree of F

symbol		Lie algebra	smallest	deg F
A_n	$n \geq 1$	sl_{n+1}	$n + 1$	NO
B_n	$n \geq 3$	so_{2n+1}	$2n + 1$	2
C_n	$n \geq 2$	sp_{2n}	$2n$	NO
D_n	$n \geq 4$	so_{2n}	$2n$	2
E_6		e_6	27	3
E_7		e_7	56	4
E_8		e_8	248	2
F_4		f_4	26	2
G_2		g_2	7	2

$$so_3 \cong sl_2, so_4 \cong sl_2 \times sl_2, so_5 \cong sp_4, so_6 \cong sl_4.$$

(4) *Reduction to smaller dimensions.*

Assumptions: M of dimension n with $\text{Gal}(M) \subset \text{SL}(V(M))$ and is connected (this is a simplifying assumption) and $(\text{gal}(M), V(M))$ irreducible.

Aim: If M is $(n - 1)$ -solvable, give an explicit “minimal” N of dimension $< n$, i.e., N is not $(-1 + \dim N)$ -solvable, such that (possibly after a finite extension of K) one has $M \in \{\{N\}\}$.

There are two cases:

$\text{gal}(M)$ is simple and $V(M)$ is not of minimal dimension.

$\text{gal}(M)$ is a product $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_s$ and $V(M)$ is a tensor product $V_1 \otimes \cdots \otimes V_s$ with V_i an irreducible representation of \mathfrak{g}_i for all i .

Recall: Let \mathfrak{g} be a semi-simple Lie algebra. There is a 'largest' connected group G with Lie algebra \mathfrak{g} . This G is simply connected. The adjoint action of G on \mathfrak{g} has a finite kernel Z , the center of G . We call G the *standard group* and G/Z the *adjoint group*. Any other connected group with Lie algebra \mathfrak{g} has the form G/Z' for some subgroup $Z' \subset Z$. Here is the list for Z and \mathfrak{g} simple:

$\mathbb{Z}/(n+1)\mathbb{Z}$ for A_n ; (SL_{n+1} and Z its center).

$\mathbb{Z}/2\mathbb{Z}$ for B_ℓ, C_ℓ, E_7 ;

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for D_ℓ with ℓ even;

$\mathbb{Z}/4\mathbb{Z}$ for D_ℓ with ℓ odd;

$\mathbb{Z}/3\mathbb{Z}$ for E_6 ;

0 for E_8, F_4, G_2 .

We will use *Fact:* If W is a faithful representation of a group H , then $Repr_H = \{\{W\}\}$.

(4.1) *Suppose that \mathfrak{g} is simple and that $Gal(M) = G$ is simply connected.*

Then one has equivalences

$\{\{M\}\} \rightarrow Repr_G \rightarrow Repr_{\mathfrak{g}}$. Further $Repr_G = \{\{V(M)\}\}$ since the representation $(G, V(M))$ is faithful. Thus $Repr_{\mathfrak{g}}$ is generated by $(\mathfrak{g}, V(M))$.

Let W be a faithful representation of \mathfrak{g} of minimal dimension. Then W is obtained by a construction of linear algebra (which can be made explicit) from $(\mathfrak{g}, V(M))$. Using the equivalences, the same construction of linear algebra applied to M produces the required minimal N with $M \in \{\{N\}\}$.

(4.2) Assume \mathfrak{g} is simple and $Gal(M) = G/Z$ where the center Z of the simply connected G is not trivial.

The category $Repr_{Gal(M)} = \{\{V(M)\}\}$ has as generator the adjoint representation Ad of G . Hence $\{\{Ad\}\} = \{\{V(M)\}\}$ and there are constructions of linear algebra from $V(M)$ to Ad and vice versa. These constructions can be found explicitly by viewing $V(M)$ and Ad as \mathfrak{g} -modules and using, say, the Lie algebra packet LiE.

The equivalence $\{\{M\}\} \rightarrow Repr_{Gal(M)}$ yields, by the same construction of linear algebra, a diff. module $[Ad]$ from M such that $V([Ad]) = (G, Ad)$. Such a differential module will be called an *adjoint differential module*.

A *standard diff. module* is defined as a differential module N such that $(Gal(N), V(N))$ is a faithful G -module of minimal dimension.

Let (G, W) be a faithful G module of minimal dimension, then the G -module $\text{End}(W)$ contains \mathfrak{g} as G -submodule. This is the adjoint representation of G .

Thus a standard diff. module N produces an adjoint diff. module M as submodule of $\text{End}(N)$.

We will make this explicit.

Under the assumption that K is a C_1 -field, one has $(N, \partial) = (K \otimes V, \partial_S)$ with V is a faithful \mathfrak{g} -module of minimal dimension. Further $\partial_S = \partial_0 + S$ where ∂_0 is the derivation defined by ∂_0 is zero on V and some $S \in K \otimes_{\mathbb{C}} \mathfrak{g} \subset \text{End}_K(N)$.

Then the adjoint module $M = K \otimes_{\mathbb{C}} \mathfrak{g} \subset \text{End}(N)$ with derivation $A \mapsto \partial_0(A) + [A, S]$, where ∂_0 is given by ∂_0 is zero on \mathfrak{g} .

Now *our problem* is to produce a standard diff. module N for a given adjoint diff. module M .

Theorem. *Suppose that K is a C_1 -field. Let M be an adjoint differential module. There exists a unique $S \in \mathfrak{g}(K)$ such that M is isomorphic to the adjoint module induced by (N, ∂_S) . Moreover, S can be obtained by a reasonable algorithm.*

Sketch. $V(M) = \mathfrak{g}$ and $[\ , \] : \Lambda^2 \mathfrak{g} \rightarrow \mathfrak{g}$ comes from a (unique up to \mathbb{C}^*) Lie algebra structure $[\ , \]_M : \Lambda^2 M \rightarrow M$ with $\partial[a, b]_M = [\partial a, b]_M + [a, \partial b]_M$.

! $[\ , \]_M$ generates the 1-dimensional \mathbb{C} -vector space $(\ker \partial, \text{Hom}(\Lambda^2 M, M))$!

The assumption K is a C_1 -field implies that there exists an isomorphism of Lie algebras $\alpha : M \rightarrow K \otimes_{\mathbb{C}} \mathfrak{g}$. In particular, M has a split Cartan subalgebra.

! A split Cartan submodule can be computed, this gives an explicit α and a ∂_0 !

The K -linear map $\partial - \partial_0$ on M is a K -linear derivation $[\cdot, S]$ of the Lie algebra M .

! The $S \in M = K \otimes \mathfrak{g}$ can be computed! \square

A *baby example* for the theorem (Fano, Singer, van Hoeij, vdP):

M of dimension 3 with Galois group PSL_2 over a C_1 -field is a $\mathrm{sym}^2 N$ with N of dimension 2 and Galois group SL_2 .

Fano's example of the introduction.

M of dimension 5 with $(\mathrm{Gal}(M), V(M))$ the standard representation of SO_5 . The Lie algebra of SO_5 is \mathfrak{sp}_4 and $\mathrm{SO}_5 = \mathrm{Sp}_4/\mu_2$ is the adjoint group.

We use now the packet Lie. Notations: the fundamental weights for \mathfrak{sp}_4 are denoted by ω_1, ω_2 . Any irreducible representation is given

by a combination $n_1\omega_1 + n_2\omega_2$ with integers $n_1, n_2 \geq 0$. One writes $[n_1, n_2]$.

The standard representation is $[1, 0]$ (dim 4), the adjoint representation of $[2, 0]$ (dim 10). M corresponds to $[0, 1]$ (dim 5) and $\Lambda^2[0, 1] = [2, 0]$. Hence $\Lambda^2 M$ is an adjoint diff. module and the above produces the standard module N of dimension 4 with $M \in \{\{N\}\}$.

There is a quicker way to compute N , namely by the observation $\Lambda^2[1, 0] = [0, 1] + [0, 0]$ and some tricks.

(5) In the following table of irreducible representations, fundamental weights $\omega_1, \dots, \omega_d$ are fixed and $[n_1, \dots, n_d]$ is the irreducible representation with weight $n_1\omega_1 + \dots + n_d\omega_d$.

*Table of the irreducible representations of
dimension $d \leq 6$.*

d	Lie alg	repr	Λ^2	sym^2
2	\mathfrak{sl}_2	[1]	[0]	[2]
3	\mathfrak{sl}_2	!!![2]	[2]	[4], [0]
3	\mathfrak{sl}_3	[1, 0]	[0, 1]	[2, 0]
4	\mathfrak{sl}_2	!!![3]	[4], [0]	[6], [2]
4	\mathfrak{sl}_4	[1, 0, 0]	[0, 1, 0]	[2, 0, 0]
4	\mathfrak{sp}_4	[1, 0]	[0, 1], [0, 0]	[2, 0]
4	$\mathfrak{sl}_2 \times \mathfrak{sl}_2$!!![1] \otimes [1]	[0] \otimes [2], [2] \otimes [0]	[0] \otimes [0], [2] \otimes [2]
5	\mathfrak{sl}_2	!!![4]	[6], [2]	[8], [4], [0]
5	\mathfrak{sp}_4	!!![0, 1]	[2, 0]	[0, 2], [0, 0]
5	\mathfrak{sl}_5	[1, 0, 0, 0]	[0, 1, 0, 0]	[2, 0, 0, 0]
6	\mathfrak{sl}_2	!!![5]	[8], [4], [0]	[10], [6], [2]
6	\mathfrak{sl}_3	!!![2, 0]	[2, 1]	[4, 0], [0, 2]
6	\mathfrak{sl}_4	!!![0, 1, 0]	[1, 0, 1]	[0, 2, 0], [0, 0, 0]
6	\mathfrak{sl}_6	[1, 0, 0, 0, 0]	[0, 1, 0, 0, 0]	[2, 0, 0, 0, 0]
6	\mathfrak{sp}_6	[1, 0, 0]	[0, 1, 0], [0, 0, 0]	[2, 0, 0]
6	$\mathfrak{sl}_2 \times \mathfrak{sl}_2$!!![1] \otimes [2]	[0] \otimes [0], [0] \otimes [4], [2] \otimes [2]	[0] \otimes [2], [2] \otimes [0], [0] \otimes [0], [2] \otimes [2]
6	$\mathfrak{sl}_2 \times \mathfrak{sl}_3$!!![1] \otimes [1, 0]	[0] \otimes [2, 0], [2] \otimes [0, 1]	[0] \otimes [0, 1], [2] \otimes [2]

For the \mathfrak{sl}_n with $n > 2$ we have omitted duals of representations. Further we have left out symmetric cases. The decompositions of the second symmetric power and the second exterior power are useful to distinguish the various cases. The items with a !!! can be expressed in terms of representations of lower dimension.

In dimensions 7 – 10, one finds for the new items of this sort (here we omit the case \mathfrak{sl}_2 and again we omit duals and symmetric situations) the list:

\mathfrak{sl}_3 with $[1, 1]$ (dim 8), $[3, 0]$ (dim 10);

\mathfrak{sl}_4 with $[2, 0, 0]$ (dim 10);

\mathfrak{sl}_5 with $[0, 1, 0, 0]$ (dim 10);

\mathfrak{so}_7 with $[0, 0, 1]$ (dim 8);

\mathfrak{sp}_4 with $[2, 0]$ (dim 10);

$\mathfrak{sl}_2 \times \mathfrak{sl}_2$ with $[1] \otimes [3]$ (dim 8); $[2] \otimes [2]$ (dim 9); $[1] \otimes [4]$ (dim 10);

$\mathfrak{sl}_2 \times \mathfrak{sl}_3$ with $[2] \otimes [1, 0]$ (dim 9);

$\mathfrak{sl}_2 \times \mathfrak{sl}_4$ with $[1] \otimes [1, 0, 0]$ (dim 8);

$\mathfrak{sl}_2 \times \mathfrak{sp}_4$ with $[1] \otimes [1, 0]$ (dim 8), with $[1] \otimes [0, 1]$ (dim 10);

$\mathfrak{sl}_2 \times \mathfrak{sl}_5$ with $[1] \otimes [1, 0, 0, 0]$ (dim 10);

$\mathfrak{sl}_3 \times \mathfrak{sl}_3$ with $[1, 0] \otimes [1, 0]$ (dim 9);

$\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ with $[1] \otimes [1] \otimes [1]$ (dim 8).

(6) *More explicit cases.*

Case \mathfrak{sl}_4 with $[0, 1, 0]$, corresponds to SL_4/μ_2 . (Note that the adjoint case is $[1, 0, 1]$ with group SL_4/μ_4).

Theorem. Let M be a differential module of dimension 6. Equivalent are:

(1) $M \cong \Lambda^2 N$ for some module of dimension 4 with $\det N = 1$.

(2) There exists $F \in \text{sym}^2 M$ with $\partial F = 0$ such that F is non degenerate and M has a totally isotropic subspace of dimension 3.

We show here the proof of $(2) \Rightarrow (1)$

By assumption F can be written in the form

$$F = m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23}$$

for a basis m_{12}, \dots, m_{34} of M . Let $(\alpha_{ij,kl})$ be the matrix of ∂ on M with respect to this basis, i.e., $\partial m_{ij} = \sum \alpha_{kl,ij} m_{kl}$.

The equality $\partial F = 0$ is equivalent to the set of equalities

$$\alpha_{ij,kl} = 0 \text{ if } \{i, j, k, l\} = \{1, 2, 3, 4\} ; \alpha_{ij,ij} + \alpha_{kl,kl} = 0 \text{ for } ij \neq kl ;$$

$$\alpha_{ik,jk} = \pm \alpha_{ik',jk'} \text{ if } \{i, j, k, k'\} = \{1, 2, 3, 4\} \text{ and}$$

the sign is $-$ for $\{i, j\} = \{1, 3\}, \{2, 4\}$ and is $+$ for the other tuples $\{i, j\}$. (Note that in the last set of relations we do not insist on $i < k, j < k$ etc.).

Consider a vector space N over K with basis n_1, \dots, n_4 , define the K -linear bijection $f : \Lambda^2 N \rightarrow M$ by sending $n_{ij} := n_i \wedge n_j$ to m_{ij} for all $1 \leq i < j \leq 4$. On N we consider an operation of ∂ given by a matrix $(\beta_{i,j})$. Now f is an isomorphism of differential modules if and only if $\beta_{i,j}$ satisfies the set of equations

$$\beta_{i,i} + \beta_{j,j} = \alpha_{ij,ij} \text{ for } 1 \leq i < j \leq 4 , \beta_{a,b} = \alpha_{aj,bj} \text{ if } a < b, a < j, b < j ;$$

$$\beta_{a,b} = -\alpha_{ai,ib} \text{ if } a < b, a < i, i < b ; \beta_{a,b} = \alpha_{ia,ib} \text{ if } a < b, i < a, i < b ;$$

$$\beta_{a,b} = \alpha_{aj,bj} \text{ if } b < a, a < j, b < j ; \beta_{a,b} = -\alpha_{ja,bj} \text{ if } b < a, j < a, b < j ;$$

$$\beta_{a,b} = \alpha_{ia,ib} \text{ if } b < a, i < a, i < b .$$

This overdetermined set of equations has a unique solution. Indeed, one finds

$$\beta_{1,1} = (\alpha_{12,12} + \alpha_{13,13} - \alpha_{23,23})/2, \quad \beta_{2,2} = (\alpha_{23,23} + \alpha_{24,24} - \alpha_{34,34})/2,$$

$$\beta_{3,3} = (\alpha_{23,23} + \alpha_{34,34} - \alpha_{24,24})/2, \quad \beta_{4,4} = (\alpha_{24,24} + \alpha_{34,34} - \alpha_{23,23})/2.$$

For each $a \neq b$ the above list gives two equations for $\beta_{a,b}$. The two equations coincide, due to the relations $\alpha_{ik,jk} = \pm \alpha_{ik',jk'}$, listed above.

Case $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ with $[1] \otimes [1]$.

theorem Let M be a differential module over K of dimension 4 with $\det M = 1$ (no further conditions on M and K). Equivalent are:

(1) M is isomorphic to $A \otimes B$ for modules A, B of dimension 2 and with $\det A = \det B = 1$

(2) There exists $F \in \text{sym}^2 M$, $\partial F = 0$, F is non degenerate and has an isotropic subspace of dimension 2.

We show here (2) \Rightarrow (1).

For a suitable basis $\{m_{ij} \mid 1 \leq i, j \leq 2\}$ one has $F = m_{11} \otimes m_{22} - m_{12} \otimes m_{21}$. We consider now K -vector spaces A and B with bases a_1, a_2 and b_1, b_2 . Define a K -isomorphism $\phi : A \otimes B \rightarrow M$ by sending $a_i \otimes b_j$ to m_{ij} for all $1 \leq i, j \leq 2$. We make A and B into differential modules by putting $\partial a_i = \sum \alpha_{j,i} a_j$ and $\partial b_i = \sum \beta_{j,i} b_j$. The two matrices $(\alpha_{i,j})$, $(\beta_{i,j})$ are as yet unknown. We only require that their traces are 0.

Let ∂ on M be given by $\partial m_{ij} = \sum \gamma_{kl,ij} m_{kl}$. The assumption $\partial F = 0$ leads to

$$\begin{aligned} \gamma_{11,22} = \gamma_{22,11} = \gamma_{12,21} = \gamma_{21,12} = 0, \quad \gamma_{11,11} + \gamma_{22,22} = \gamma_{12,12} + \gamma_{21,21} = 0, \\ \gamma_{12,11} = \gamma_{22,21}, \quad \gamma_{21,11} = \gamma_{22,12}, \quad \gamma_{12,22} = \gamma_{11,21}, \quad \gamma_{21,22} = \gamma_{11,12} . \end{aligned}$$

The assumption that ϕ is an isomorphism of differential modules leads to a unique solution for the matrices $(\alpha_{i,j})$, $(\beta_{i,j})$, namely

$$\alpha_{1,2} = \gamma_{11,21} = \gamma_{12,22}, \quad \alpha_{2,1} = \gamma_{22,12} = \gamma_{21,11} ,$$

$$\alpha_{11} = (\gamma_{11,22} + \gamma_{12,12})/2, \quad \alpha_{22} = (\gamma_{21,21} + \gamma_{22,22})/2,$$

$$\beta_{1,2} = \gamma_{11,12} = \gamma_{21,22}, \quad \beta_{2,1} = \gamma_{22,21} = \gamma_{12,11},$$

$$\beta_{11} = (\gamma_{11,22} + \gamma_{21,21})/2, \quad \beta_{22} = (\gamma_{12,12} + \gamma_{22,22})/2.$$