Analytic q-difference equations, universal rings and universal Galois groups.
(joint work with Marc Reversat)

Notation: $q \in \mathbf{C},\ 0 < |q| < 1,\ q = e^{2\pi i\tau}$, $im(\tau) > 0,\ q^{\lambda} := e^{2\pi i\tau\lambda}$

Difference field F, provided with an automorphism ϕ of infinite order.

Ex: $K = \mathbf{C}(\{z\}), \widehat{K} = \mathbf{C}((z))$ and their algebraic closures $\overline{K},\ \overline{\widehat{K}}$. Everywhere $\phi(z^{\lambda}) = q^{\lambda}z^{\lambda}$.

Difference module $M = (M, \Phi)$, $\dim_F M < \infty$, $\Phi : M \to M$ additive, bijective and $\Phi(fm) = \phi(f)\Phi(m)$. *Solution*: $m \in M$ with $\Phi(m) = m$.

Can be converted into a *matrix* difference equation $\quad y(z) = A(z)y(qz),\ A \in \mathrm{GL}_n(F)$
and into a *scalar* difference equation, like
$y(q^2z) + a_1y(qz) + a_0y(z) = 0,\ a_0, a_1 \in F$

Regular singular modules

M is called *regular singular* if M contains a $\mathbf{C}\{z\}$-lattice, invariant under Φ, Φ^{-1}. By the following constructions of linear algebra:

sum: $M_1 \oplus M_2$ with $\Phi(m_1, m_2) = (\Phi m_1, \Phi m_2)$,

tensor product: $M_1 \otimes M_2 = M_1 \otimes_F M_2$ with $\Phi(m_1 \otimes m_2) = (\Phi m_1) \otimes (\Phi m_2)$,

they are obtained from the two examples:

(a) $Ke, \Phi(e) = c^{-1}e,\ c \in \mathbf{C}^*$.

Symbolic solution $e(c) \cdot e$ with interpretation z^d where $d = \frac{\log c}{2\pi i \tau}$, for $e(c)$.

(b) $U_n := Ke_1 + \cdots + Ke_n$ with Φ given by the matrix

$$\begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & & 1 \\ & & & 1 \end{bmatrix}$$

is called *unipotent* of length n. All solutions live in $K[\ell]$ with $\phi(\ell) = \ell + 1$. Interpretation of ℓ is $\frac{\log z}{2\pi i \tau}$.

Irregular modules
Ke, $\Phi e = z^{-t}e$, with integer $t \neq 0$, is *irregular*; Symbolic solution $e(z^t) \cdot e$. Meromorphic interpretation on $\mathbf{C}^*$ of $e(z^t)$ is $\Theta(-z)^{-t}$ with $\Theta := \sum_{n \in \mathbf{Z}} q^{n(n-1)/2}(-z)^n$.
Indeed, $(-z)\Theta(qz) = \Theta(z)$.

Towards a classification of difference modules over K.
G.G. Birkhof, P.E. Guenther, C.R. Adams, J.-P. Ramis, Ch. Zhang, J. Sauloy, A. Duval et al.
Ingredients: $K[\Phi, \Phi^{-1}]$, the skew ring of difference operators, is Euclidean.
Any difference module M is *cyclic*, i.e., $M \cong K[\Phi, \Phi^{-1}]/K[\Phi, \Phi^{-1}]L$ with
$L = \Phi^m + a_{m-1}\Phi^{m-1} + \cdots + a_1\Phi + a_0,\ a_0 \neq 0.$

The *Newton polygon* of M is obtained from the order of the coefficients a_i of L. The slopes of the Newton polygon are in $\mathbf{Q}$.

Definition: M is *pure* if there is only one slope.
Examples:
M is regular singular $\Leftrightarrow$ M is pure of slope 0.

$Ke, \ \Phi e = cz^t e, \ c \in \mathbf{C}^*$ is pure of slope t.

Definition of $E(cz^{t/n}) = (K_n e, \ \Phi e = cz^{t/n} e)$, with $n \geq 1, \ (t,n) = 1, \ c \in \mathbf{C}^*, |q|^{1/n} < |c| \leq 1$, is considered as difference module over K. Then $E(cz^{t/n})$ has dimension n over K and slope t/n.

The classification of the pure difference modules over K follows from:

Thm. The indecomposable pure modules over K are $E(cz^{t/n}) \otimes U_m$ with $|q|^{1/n} < |c| \leq 1$. Further$(t/n, c^n, m)$ is unique.

This is ‘equivalent’ to the Atiyah’s classification of the indecomposable vector bundles on the *elliptic curve* $E_q = \mathbf{C}^*/q^{\mathbf{Z}}$.

The slope filtration and moduli spaces

Theorem (Birkhoff, Adams, Ramis, Sauloy) M has a unique tower of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ such that every M_i/M_{i-1} is pure of slope λ_i and $\lambda_1 < \cdots < \lambda_r$.

One writes $gr(M) := \oplus M_i/M_{i-1}$.

Choose pure modules P_i of slope λ_i with $\lambda_1 < \cdots < \lambda_r$ and put $S := P_1 \oplus \cdots \oplus P_r$.

Consider pairs (M, f) with $f : gr(M) \to S$ an isomorphism. Two pairs $(M(1), f_1), (M(2), f_2)$ are equivalent if there exists an isomorphism $g : M(1) \to M(2)$ with $f_1 = f_2 \circ gr(g)$.

Thm. There exists a *fine moduli space* for the collection of equivalence classes of the pairs (M, f). This moduli space is isomorpic to $\mathbf{C}^N$ with $N = \sum_{i<j}(\lambda_j - \lambda_i) \dim P_i \cdot \dim P_j$.

Basic example, present in the work of G. Birkhoff.
$P_1 = Ke_1, \Phi e_1 = e_1$,
$P_2 = Ke_1,\ \Phi e_2 = (-z)^t e_2$ and $t > 0$.
The moduli space is $\mathbf{C}^t$ and the universal family above this moduli space is

$$K[x_0, \ldots, x_{t-1}]e_1 + K[x_0, \ldots, x_{t-1}]e_2,\ \Phi e_1 = e_1,$$

$$\Phi e_2 = (-z)^t e_2 + (x_0 + x_1 z + \cdots + x_{t-1} z^{t-1})e_1\ .$$

Remark: Over the field $\widehat{K} = \mathbf{C}((z))$ every q-diff.module is direct sum of pure modules. For a q-diff.module M over K the decomposition of $\widehat{K} \otimes M$ into pure modules introduces divergent power series. Example: $f = \sum_{n \geq 1} q^{-n(n+1)/2} z^n$, solution of $(z^{-1}\phi - 1)f = 1$. This plays an important role for the (universal) difference Galois group.

Difference Galois groups

A q-diff. module M has, as in the differential case, a *Picard-Vessiot ring* $PV \supset K$, a solution space $Sol(M) = \ker(\Phi - 1, PV \otimes M)$ of finite dimension over $\mathbf{C}$ and a *difference Galois group* $Gal(M)$ consisting of the K-automorphisms of PV commuting with ϕ. The group $Gal(M)$ acts faithfully on $Sol(M)$ and is identified with an algebraic subgroup of $\mathrm{GL}(Sol(M))$.

M is called *split* if M is a direct sum of pure modules. For the indecomposable pure module $M := E(cz^{t/n}) \otimes U_m$ the difference Galois group is explicitly known. Indeed, there is an exact sequence (not semi-direct)

$$1 \to \mathbf{C}^*(\times \mathbf{C}) \to Gal(M) \to (\mathbf{Z}/n\mathbf{Z})^2 \to 0 \ .$$

For general M the module $S := gr(M)$ is split and M corresponds to a point ξ in the moduli space corresponding to S. There is an exact sequence

$$1 \to U_\xi \to Gal(M) \to Gal(S) \to 1,$$

where U_ξ is a computable unipotent group depending on ξ with known action of $Gal(S)$.
Conclusion: $Gal(M)$ is computable.
Remark: Unlike the differential case, $Gal(M)^o$ is solvable!

Universal Picard-Vessiot rings and the universal difference Galois groups.

For a Tannakian category of q-difference equations one can form a (universal) Picard-Vessiot ring U and a difference Galois group, which is an affine group scheme and corresponds to the automorphisms of U commuting with ϕ.

(1) *All regular singular modules over* K. Its PVR is $Univ_{rs} = \overline{K}[\{e(c)\}_{c\in\mathbf{C}^*}, \ell]$ with rules:

$$e(c_1c_2) = e(c_1)\cdot e(c_2),\ e(q^\lambda) = z^{-\lambda} \text{ for rational } \lambda$$

$$\text{and } \phi(e(c)) = c^{-1}e(c),\ \phi(\ell) = 1 + \ell$$

The corresponding difference Galois group G_{rs} is $\mathrm{Hom}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*) \times \mathbf{C}$.

(2) *All split q-diff. modules over* K has PVR

$$Univ_{split} = \overline{K}[\{e(c)\}_{c\in\mathbf{C}^*}, \ell, \{e(z^\lambda)\}_{\lambda\in\mathbf{Q}}]$$

with additional rules

$$e(z^{\lambda+\mu}) = e(z^\lambda)\cdot e(z^\mu),\ \phi(e(z^\lambda)) = z^{-\lambda}\cdot e(z^\lambda)$$

Its difference Galois group G_{split} admits an exact sequence

$$1 \to \mathrm{Hom}(\mathbf{Q}, \mathbf{C}^*) \to G_{split} \to G_{rs} \to 1$$

The group scheme G_{split} is *not* a semi-direct product and $\mathrm{Hom}(\mathbf{Q}, \mathbf{C}^*)$ lies in the center of G_{split}. This contrasts the differential case!

(3) Δ_K = *All q-difference modules over* K. The universal PVR is $Univ = \mathcal{D}[\{e(c)\}, \ell, \{e(z^\lambda)\}]$ where $\mathcal{D}$ is the $\overline{K}$-subalgebra of $\widehat{\overline{K}}$ consisting of the elements $f \in \widehat{\overline{K}}$ satisfying a scalar q-differential equation over $\overline{K}$.
$\mathcal{D}$ is generated as a $\overline{K}$-algebra by the solutions in $\widehat{\overline{K}}$ of all equations of the form

$$(z^{-\lambda_1}\phi - 1)^{m_1} \cdots (z^{-\lambda_r}\phi - 1)^{m_r} f = 1 \ ,$$

where $0 < \lambda_1 < \cdots < \lambda_r,\ r \geq 1,\ m_1, \ldots, m_r \geq 1$.

The difference Galois group G_{conv} admits an exact sequence (semi-direct product)

$$1 \to N \to G_{conv} \to G_{split} \to 1$$

and N is a connected locally unipotent group scheme determined by its (pro-) Lie algebra $Lie(N)$ consisting of the $\overline{K}$-linear derivations $D : \mathcal{D} \to Univ$ commuting with ϕ. If $Lie(N)$ is known in detail then also G_{conv} would be known. The obstruction is the absence of an explicit description of the algebra $\mathcal{D}$.

Now we present the *intermediate Tannakian category* $\Delta_{2,K} \subset \Delta_K$, generated by the difference modules having at most two slopes. One has $PVR(\Delta_{2,K}) = \mathcal{D}_2[\{e(c)\}, \ell, \{e(z^\lambda)\}]$ for a certain $\overline{K}$-subalgebra $\mathcal{D}_2 \subset \mathcal{D} \subset \widehat{\overline{K}}$ and a universal difference Galois group G_2 which is the semi-direct product of G_{split} and a (connected) unipotent group scheme N_2. The latter is a quotient of N and the pro-Lie algebra $Lie(N_2)$ is a quotient of $Lie(N)$. Any $D \in Lie(N_2)$ is a $\overline{K}$-linear derivation $D : \mathcal{D}_2 \to PVR(\Delta_{2,K})$, commuting with ϕ.

The elements $f_{m,c,\mu}$.
For $\mu \in \mathbf{Q}$ with $\mu > 0$, $c \in \mathbf{C}^*$ with $|q^\mu| < |c| \leq 1$ and $m \geq 1$, the unique solution in $\widehat{\overline{K}}$ of the equation $(c^{-1}z^{-\mu}\phi - 1)^m y = 1$ is called $f_{m,c,\mu}$.

Theorem: $\mathcal{D}_2$ is generated over $\overline{K}$ by the elements $f_{m,c,\mu}$. These elements are algebraically independent over $\overline{K}$.

The pro-Lie algebra $Lie(N_2)$ is commutative and the theorem yields an explicit topological basis $\{D_{\mu,c,n}\}$. The 'two slopes' in the definition of $\Delta_{2,K}$ correspond to 'divergence with one level'. For the latter, J.-P. Ramis and J. Sauloy have results on q-summation which seem to agree with these topological generators.

Finally, we *conjecture* that $Lie(N_2)$ is actually $Lie(N)_{ab} := Lie(N)/[Lie(N), Lie(N)]$ and moreover, the topological generators $\{D_{\mu,c,n}\}$ can be lifted to topological generators for $Lie(N)$.