Symbolic-numeric Computation of Implicit Riquier Bases for PDE

Wenyuan Wu

joint work with Greg Reid

Department of Applied Mathematics, UWO.

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Background and Motivation

- Exact prolongation-elimination algorithms for exact polynomially nonlinear PDE are well studied in [1, 3, 4, 9, 6, 7]. Identify all hidden constraints and compute formal power series solutions in the neighborhood of a given point.

  Example 1 (The Pendulum) *For the pendulum of unit mass, under constant gravity, we have*

  $$\begin{aligned} X_{tt} + \lambda X &= 0 \\ Y_{tt} + \lambda Y &= -g \\ X^2 + Y^2 &= 1. \end{aligned} \tag{1}$$

  We need to differentiate the third eqn. twice to reduce this DAE to ODE. So for such differential systems, differentiation (Prolongation) is unavoidable.

- However a major problem in these approaches is the exploding size of prolongations for many independent variable, which cause a huge nonlinear system with dramatically large Bezout number (measure difficulty of solving algebraic system).

One PDE:

J^1

$u_x + v^3 = 0$

Prolongation once:

$$\begin{tikzcd}
J^1 \arrow[r, shift left, "D"] & J^2 \arrow[l, shift left, "\pi"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & u_{xx}+3v^2v_x=0 \\
u_x+v^3=0 \arrow[ur, "\frac{\partial}{\partial x}"] \arrow[dr, "\frac{\partial}{\partial y}"'] & \\
 & u_{xy}+3v^2v_y=0
\end{tikzcd}$$

Prolongation twice:

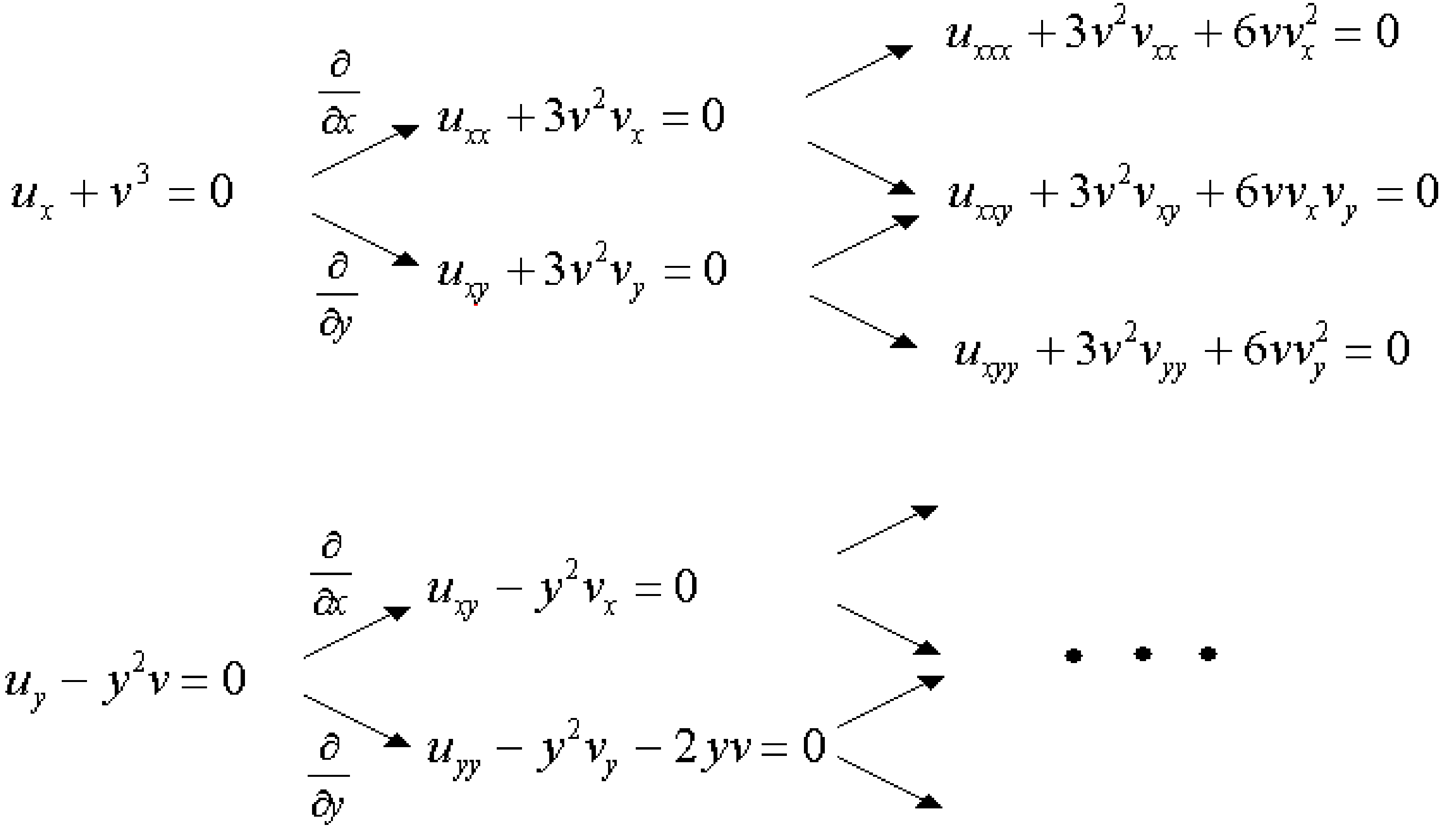
$$\begin{tikzcd}
J^1 \arrow[r, shift left, "D"] & J^2 \arrow[l, shift left, "\pi"] \arrow[r, shift left, "D"] & J^3 \arrow[l, shift left, "\pi"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & & u_{xxx}+3v^2v_{xx}+6vv_x^2=0 \\
 & u_{xx}+3v^2v_x=0 \arrow[ur] \arrow[dr] & \\
u_x+v^3=0 \arrow[ur, "\frac{\partial}{\partial x}"] \arrow[dr, "\frac{\partial}{\partial y}"'] & & u_{xxy}+3v^2v_{xy}+6vv_xv_y=0 \\
 & u_{xy}+3v^2v_y=0 \arrow[ur] \arrow[dr] & \\
 & & u_{xyy}+3v^2v_{yy}+6vv_y^2=0
\end{tikzcd}$$

total degree does not change, Bezout # increase exponentially.

Two PDEs:

$$\begin{tikzcd} J^1 \arrow[r, shift left, "D"] & J^2 \arrow[l, shift left, "\pi"] \arrow[r, shift left, "D"] & J^3 \arrow[l, shift left, "\pi"] \end{tikzcd}$$



$$u_x + v^3 = 0$$

$\frac{\partial}{\partial x}$: $$u_{xx} + 3v^2 v_x = 0$$

$\frac{\partial}{\partial y}$: $$u_{xy} + 3v^2 v_y = 0$$

$$u_{xxx} + 3v^2 v_{xx} + 6vv_x^2 = 0$$

$$u_{xxy} + 3v^2 v_{xy} + 6vv_x v_y = 0$$

$$u_{xyy} + 3v^2 v_{yy} + 6vv_y^2 = 0$$

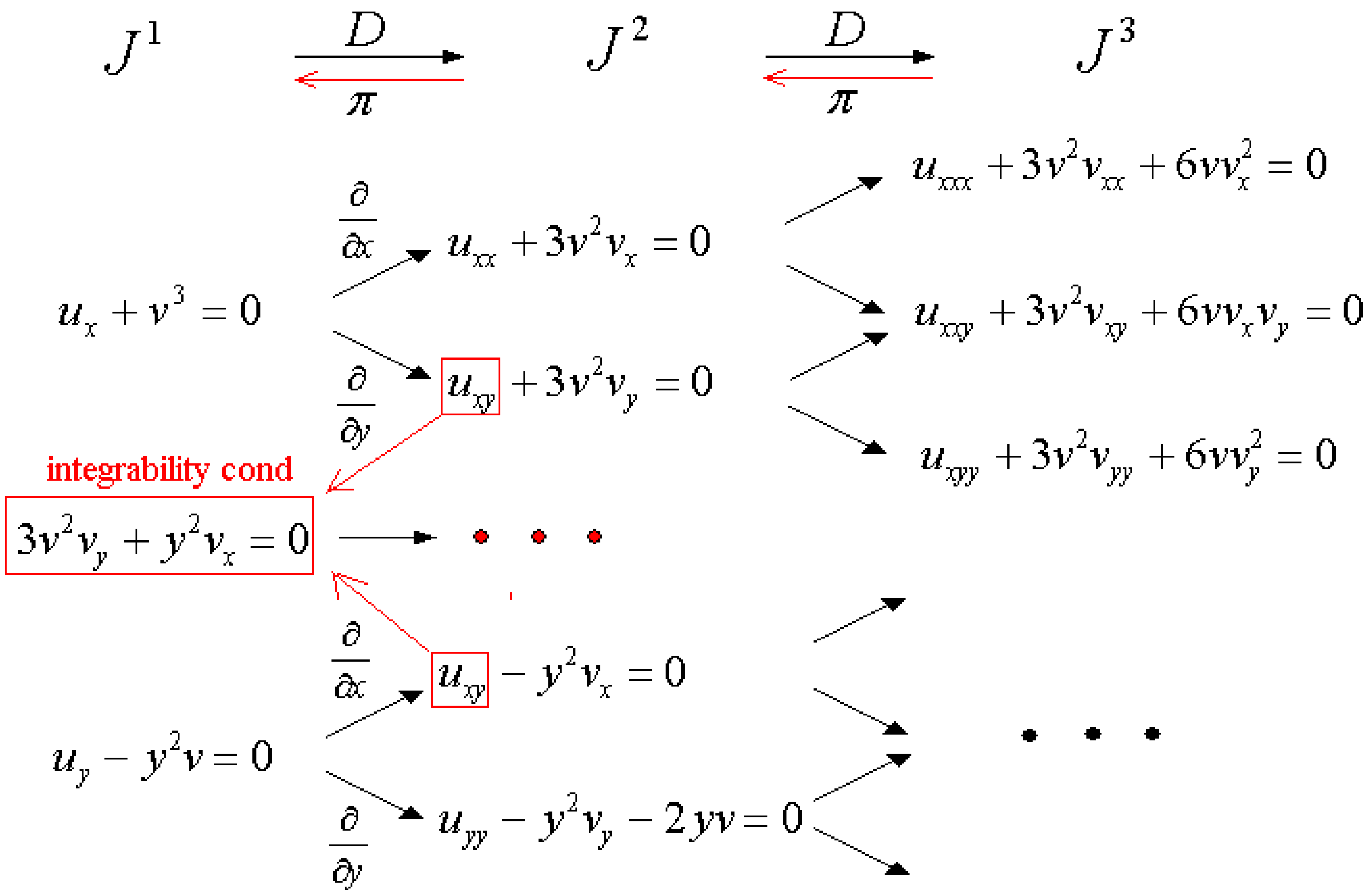
$$u_y - y^2 v = 0$$

$\frac{\partial}{\partial x}$: $$u_{xy} - y^2 v_x = 0$$

$\frac{\partial}{\partial y}$: $$u_{yy} - y^2 v_y - 2yv = 0$$

• • •

Integrability Cond.



$$J^1 \underset{\pi}{\overset{D}{\rightleftarrows}} J^2 \underset{\pi}{\overset{D}{\rightleftarrows}} J^3$$

$$u_x + v^3 = 0 \quad \xrightarrow{\frac{\partial}{\partial x}} \quad u_{xx} + 3v^2 v_x = 0$$

$$u_x + v^3 = 0 \quad \xrightarrow{\frac{\partial}{\partial y}} \quad \boxed{u_{xy}} + 3v^2 v_y = 0$$

$$u_{xxx} + 3v^2 v_{xx} + 6vv_x^2 = 0$$

$$u_{xxy} + 3v^2 v_{xy} + 6vv_x v_y = 0$$

$$u_{xyy} + 3v^2 v_{yy} + 6vv_y^2 = 0$$

integrability cond

$$\boxed{3v^2 v_y + y^2 v_x = 0} \longrightarrow \cdots$$

$$u_y - y^2 v = 0 \quad \xrightarrow{\frac{\partial}{\partial x}} \quad \boxed{u_{xy}} - y^2 v_x = 0$$

$$u_y - y^2 v = 0 \quad \xrightarrow{\frac{\partial}{\partial y}} \quad u_{yy} - y^2 v_y - 2yv = 0$$

$$\cdots$$

might need more Prolongation to cover all the integrability conditions, but differential elimination is more complicated!

- We give a hybrid method [7] to process the nonlinear part by Homotopy Continuation Methods without symbolic elimination of nonlinear systems e.g. Gröbner bases or triangular decomposition.

- Techniques which are helpful for the symbolic case are often unstable for the approximate case, since elimination depends on a given rankings. A numerical elimination method is developed in our ISSAC'06 paper [13] using SVD and Numerical Algebraic Geometry [10]. But only suitable for small systems.

- How to find a numerically stable method to identify all hidden constraints without prolongation explosion and complicated diff. elim.?

Key Ideas

Obviously, for arbitrary systems this is unavoidable. So we identify a certain class of PDE (called square t-dominated systems).

- only prolongations with respect to one independent variable are needed, so the number of new equations after prolongation will not change!
- no elimination is needed, only check some simple criterions for termination and failure of the algorithm .
- connection to Riquier Basis which is a differential analogue of Gröbner Basis for polynomial equations and giving algebraic interpretation of the output which enables EU theorem.
- genericity of the input and genericity of success of the algorithm.

PDE in Jet Space

Consider q-th order PDE system $R = (R^1, ..., R^\ell) = 0$ with indep vars $x = (x_1, x_2, ..., x_n)$ and dep vars $u = (u^1, u^2, ..., u^m)$ in a field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). Consider a set of indeterminates $\Omega = \{v^i_\alpha \mid \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n, i = 1, \cdots, m\}$ where each member of Ω corresponds to a partial derivative by:

$$v^i_\alpha \leftrightarrow \mathbf{D}^\alpha u^i(x_1, \cdots, x_n) := (\mathbf{D}_{x_n})^{\alpha_n} \cdots (\mathbf{D}_{x_1})^{\alpha_1} u^i(x_1, \cdots, x_n) \ .$$

The total derivative $\mathbf{D}_{x_i}$ act on functions of $\{x\} \cup \Omega$ by:

$$\mathbf{D}_{x_i} = \frac{\partial}{\partial x_i} + \sum_{v \in \Omega} (\mathbf{D}_{x_i} v) \frac{\partial}{\partial v} \qquad (2)$$

The PDE system R is associated with a locus of points

$$Z(R) := \{(x, v^i_\alpha) \in J^q(\mathbb{F}^n, \mathbb{F}^m) : R^k(x, v^i_\alpha) = 0, k = 1, ..., \ell\} \qquad (3)$$

where $J^q(\mathbb{F}^n, \mathbb{F}^m) \simeq \mathbb{F}^n \times \mathbb{F}^m \times \mathbb{F}^{m_1} \times ... \times \mathbb{F}^{m_q}$ is the jet space of order q and $m_r := m \cdot \binom{r+n-1}{r}$.

Example 2 (The Pendulum)

$$\begin{array}{rcl} X_{tt} + \lambda X & = & 0 \\ Y_{tt} + \lambda Y & = & -g \\ X^2 + Y^2 & = & 1. \end{array} \tag{4}$$

Here

$$Z(R) = \{(t, X, Y, \lambda, X_t, Y_t, \lambda_t, X_{tt}, Y_{tt}, \lambda_{tt}) \in J^2 : \\ X_{tt} + \lambda X = 0, Y_{tt} + \lambda Y + g = 0, X^2 + Y^2 - 1 = 0\}$$

is a 7 *dimensional submanifold of* $J^2 \simeq \mathbb{F}^{10}$.

Ranking

We introduce ranking here only for theory and algebraic interpretation. In computation we use implicit form without elimination, so it is stable.

Definition 1 (Ranking [8]) *A positive ranking $\prec$ of Ω is a total ordering on Ω which satisfies:*

$$v_\alpha^i \prec v_\beta^j \Rightarrow v_{\alpha+\gamma}^i \prec v_{\beta+\gamma}^j \tag{5}$$

$$v_\alpha^i \prec v_{\alpha+\gamma}^i \tag{6}$$

First we introduce a map ψ from Ω to $\mathbb{Z}^{m+n}$:

$$\psi : \frac{\partial^{\alpha_1+\cdots+\alpha_n} u^j}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \mapsto (0, \cdots, 0, 1, 0, \cdots, 0, \alpha_1, \cdots, \alpha_n)^t \tag{7}$$

where the "1" appears in jth coordinate.

We use a matrix representation following Riquier and Rust [8].

Definition 2 (Ranking by Matrix) *Suppose M is an $l \times (m+n)$ matrix with nonnegative integer entries and satisfies $\theta \neq \tau$ implies $M \cdot \psi(\theta) \neq M \cdot \psi(\tau)$. We define $\prec_M$ to be a ranking with respect to M, if $\theta, \tau \in \Omega$, we have $\theta \prec_M \tau \Leftrightarrow M \cdot \psi(\theta) < M \cdot \psi(\tau)$. Here M called a matrix representation of this ranking. And $\theta \preceq_M \tau$ means $\theta \prec_M \tau$ or $\theta = \tau$.*

An ordering of the elements in $\mathbb{Z}^l$ denoted by $<$ is defined by lexical order (comparing the values at the first coordinate, then the second coordinate, and so on).

Example 3 *If we choose M to be identity matrix, then the ranking is the elimination ranking (e.g. $v^2_{(k,0)} \prec v^1_{(0,1)}$ and $v^i_{(0,k)} \prec v^i_{(1,0)}$ for any $k \in \mathbb{N}$).*

Signature Matrix of t-Dominated Systems

Start from a simple case: two indep vars (t, x). For each u^j, we choose a ranking (only need this partial ranking in computation):

$$u^j \prec u^j_x \prec u^j_{xx} \prec \cdots \prec u^j_t \prec u^j_{tx} \prec \cdots \tag{8}$$

We hide the details about the differential order of x by defining a weight map $\varphi : \Omega \to \mathbb{R}$ as follows:

$$\varphi(v^i_\alpha) := \begin{cases} \alpha_1, & \text{if } \alpha_p = 0, \text{ for any } p \neq 1 \ ; \\ \alpha_1 + \epsilon, & \text{if there exists } p \neq 1,\ \alpha_p \neq 0 \ . \end{cases} \tag{9}$$

where α_1 is the diff. order w.r.t. t and $\epsilon > 0$ but very close to zero.

Determine the leading derivative for each equation R_i w.r.t. each u^j using the ranking (8), denoted by $\textsc{ld}(R_i, u^j)$.

Define the *signature matrix* of R (see Pryce [5] for ODE case) by

$$(\sigma_{i,j})(R) := \begin{cases} \varphi(\textsc{ld}(R_i, u^j)), & \text{if } R_i \text{ depends on } u^j \ ; \\ -\infty, & \text{otherwise} \ . \end{cases} \tag{10}$$

And define the *leading class* of derivatives $\textsc{lcd}(R) := \{\textsc{ld}(R, u^j)\}$, $1 \leq j \leq m$.

Definition 3 *We say R is dominated by pure derivatives in the independent variable t if there is no ϵ appearing in $(\sigma_{i,j})(R)$. For notational simplicity, we also call R a t-dominated system.*

Generalizing Pryce's Method to PDE

Let R be a square t-dominated system. Now if we consider R as ODE (the only independent variable is t). Suppose R_i needs to be differentiated c_i times ($c_i \geq 0$) to find all the hidden constraints. The new system after differentiation is denoted by $\mathbf{D}_t^c R$.

Suppose the highest order of u^j appear in $\mathbf{D}_t^c R$ is d_j. From the definition of $(\sigma_{i,j})$, clearly d_j is the largest of $c_i + \sigma_{ij}$, which implies that $d_j - c_i \geq \sigma_{ij}, \quad$ for all i, j.

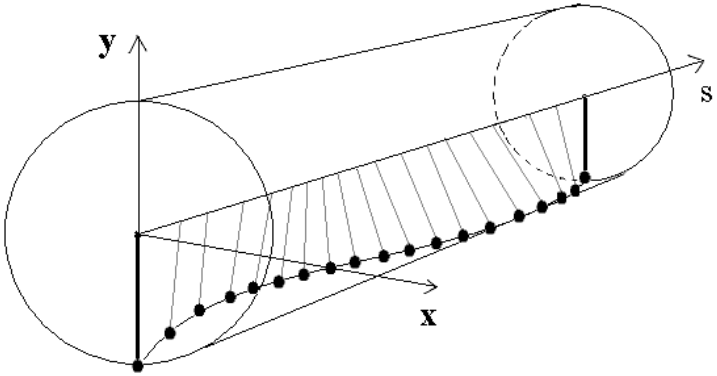
There are $\sum d_j + m$ jet variables and $\sum c_i + m$ equations in $\mathbf{D}_t^c R$ (only count pure t-derivatives). If each equation drops the dimension of the zero set by one, then the dimension of $\mathbf{D}_t^c R$ is $\sum d_j - \sum c_i$. To find all the constraints means to minimize the dimension of $\mathbf{D}_t^c R$ (Geometric Interpretation).

This can be formulated as an integer linear programming problem (LLP) in the variables $c = (c_1, ..., c_m)$ and $d = (d_1, ..., d_m)$:

$$\left\{ \begin{array}{rl} \text{Minimize} & z = \sum d_j - \sum c_i, \\ \text{where} & d_j - c_i \geq \sigma_{ij}, \\ & c_i \geq 0 \end{array} \right. \tag{11}$$

The computation of c and d which only involves the information on differential order is consequently very fast.

Example 4 *Consider a curtain made of many pendula hanging under gravity g as shown in Figure below.*



The system is:

$$
\begin{aligned}
X_{tt} + \lambda X &= \kappa X_{ss} && (12)\\
Y_{tt} + \lambda Y + g &= \kappa Y_{ss} && (13)\\
\Phi = \frac{1}{2}(X^2 + Y^2 - 1) &= 0 && (14)
\end{aligned}
$$

when $\kappa = 0$ this reduces to the simple pendulum.

It is t-dominated (and also s-dominated)and the signature matrix is:

$$\begin{pmatrix} 2 & -\infty & 0 \\ -\infty & 2 & 0 \\ 0 & 0 & -\infty \end{pmatrix}.$$ *Then LPP is*

$$\begin{cases} \textit{Minimize} & z = d_1 + d_2 + d_3 - c_1 - c_2 - c_3, \\ \textit{where} & d_1 - c_1 \geq 2, \quad d_1 - c_2 \geq -\infty, \quad d_1 - c_3 \geq 0, \\ & d_2 - c_1 \geq -\infty, \quad d_2 - c_2 \geq 2, \quad d_2 - c_3 \geq 0, \\ & d_3 - c_1 \geq 0, \quad d_3 - c_2 \geq 0, \quad d_3 - c_3 \geq -\infty, \\ & c_1 \geq 0, \quad c_2 \geq 0, \quad c_3 \geq 0 \end{cases}$$

Solving this integer LPP by LPSolve *in the Optimization package of Maple10, we obtain*

$$c_1 = 0, \quad c_2 = 0, \quad c_3 = 2; \tag{15}$$

$$d_1 = 2, \quad d_2 = 2, \quad d_3 = 0. \tag{16}$$

Jacobian Criterion for Termination

Assume $c_1 \geq c_2 \geq \cdots \geq c_m$, and let $k_c = c_1$. Then we can partition $\mathbf{D}_t^c R$ into $k_c + 1$ parts.

B_0	B_1	$\cdots$	B_{k_c-1}	B_{k_c}
$R_1^{(0)}$	$R_1^{(1)}$	$\cdots$	$R_1^{(c_1-1)}$	$R_1^{(c_1)}$
	$R_2^{(0)}$	$\cdots$	$R_2^{(c_2-1)}$	$R_2^{(c_2)}$
		$\vdots$	$\vdots$	$\vdots$
		$R_m^{(0)}$	$\cdots$	$R_m^{(c_m)}$

Table 1: The triangular block structure of $\mathbf{D}_t^c R$.

For each $B_i, 0 \leq i \leq k_c$, let $U_i := \textsc{lcd}(B_i)$ and define the *Jacobian Matrix*

$$\mathrm{Jac}_i := \left(\frac{\partial B_i}{\partial U_i} \right). \tag{17}$$

To apply the Riquier Existence Theorem, we need to refine the partial ranking (8) to a positive ranking.

Proposition 1 *Let* $\mathrm{LCD}(R) = \{\theta_1, ..., \theta_m\}$ *and let* B *be the set of all the other derivatives of* R*. Then there exists a positive ranking* $\prec$ *which satisfies the partial ranking (8) and* $\theta_1 \prec \theta_2 \prec \cdots \prec \theta_m$ *and each* θ_i *is greater than any* $b \in B$*.*

By Implicit Function Theorem and properties of analytic functions, we can show the Algebraic Interpretation:

Theorem 1 (Jacobian Criterion) *Let* R *be a square analytic t-dominated system of* PDE *and* $\boldsymbol{D}_t^c R$ *be the system of t-prolongation by solving LPP (11). If* Jac_{k_c} *is nonsingular at some point* p *in* $Z(\boldsymbol{D}_t^c R)$*, then* $\boldsymbol{D}_t^c R$ *is an implicit Riquier Basis at* p *with an order given by Proposition 1.*

Genericities

Genericity of t-dominated Systems A generic $\mathbb{F}$-analytic or polynomially nonlinear PDE system R with order k is t-dominated. Any $\mathbb{F}$-analytic or polynomially nonlinear PDE system R with order k is t-dominated after a random linear coordinates change in the independent variables with coefficients in $\mathbb{F}$.

Genericity of non-singular Jacobian For a square polynomially t-dominated PDE system R, if it is determined and the coefficient of each term is generic, then at a generic point of $\mathbf{D}_t^c R$ in Jet space, the Jacobian matrix Jac_{k_c} is non-singular.

Random linear coordinates change will destroy the sparsity, which is very bad for elimination! But here no elimination and just one direction prolongation, the cost will not increase too much.

Experimental Results

The t-prolongation procedure for ODE and PDE was implemented in Maple10 with polynomial solver PHCPack [11]. The integer linear programming involved using Maple10's LPSolve command. We applied our maple program to a Test Set of Visconti [12] containing 27 DAE with index ranging from 1 to 6.

- The procedure identified index consistent with Visconti's results.
- The LP problems were solved in less than one second.
- Our 6 failures were due to: 3 non-square system; 3 systems with singular Jacobians.
- Like other standard DAE approaches, Visconti required an initial guess for a consistent initial point, but we use homotopy continuation method to compute initial points.

Detailed Examples

Example 5 (ODE for a Crane, Index 5) *This model which is illustrated in Figure 1, is discussed in [2]. The problem is to determine the horizontal velocity* $u_1(t)$ *of a winch of mass* M_1*, and the angular velocity* $u_2(t)$ *of the winch so that the attached load* M_2 *moves along a prescribed path.*

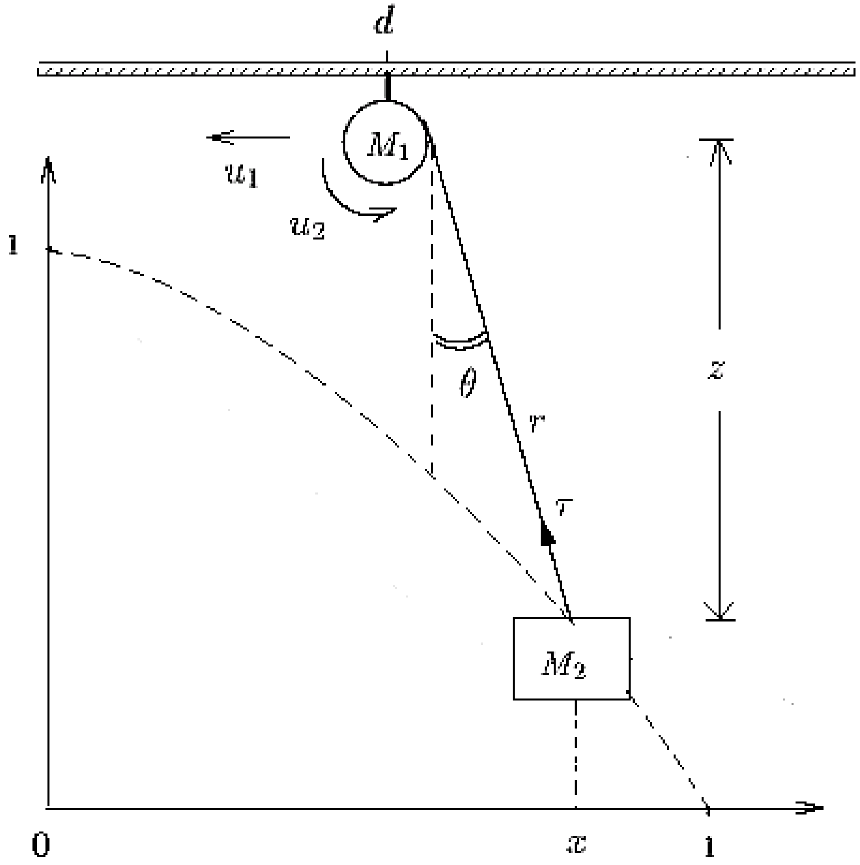
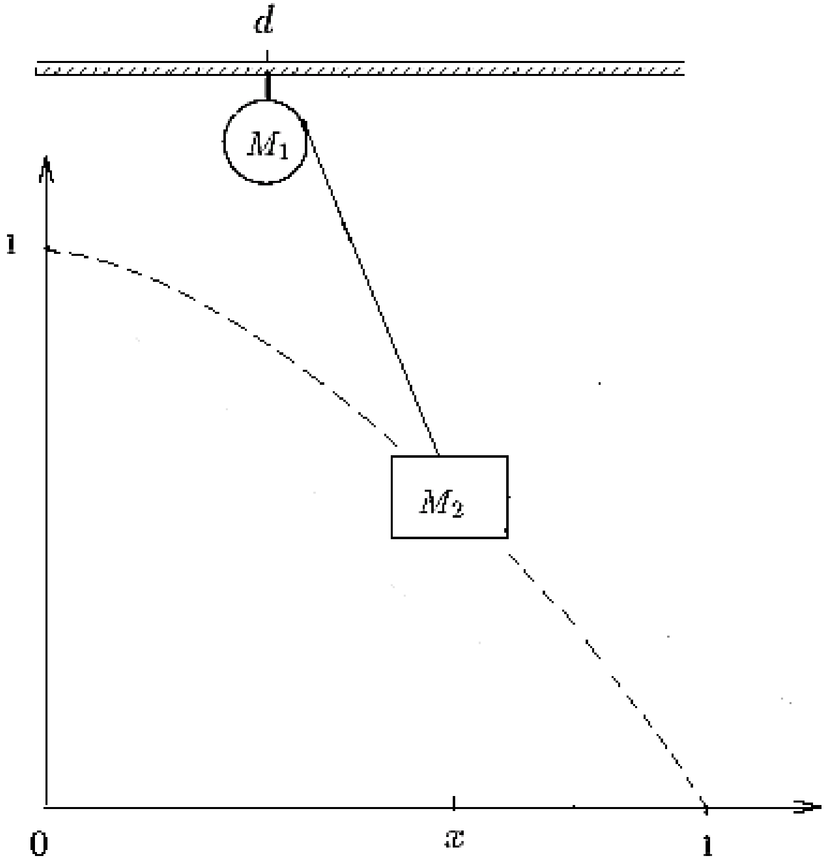
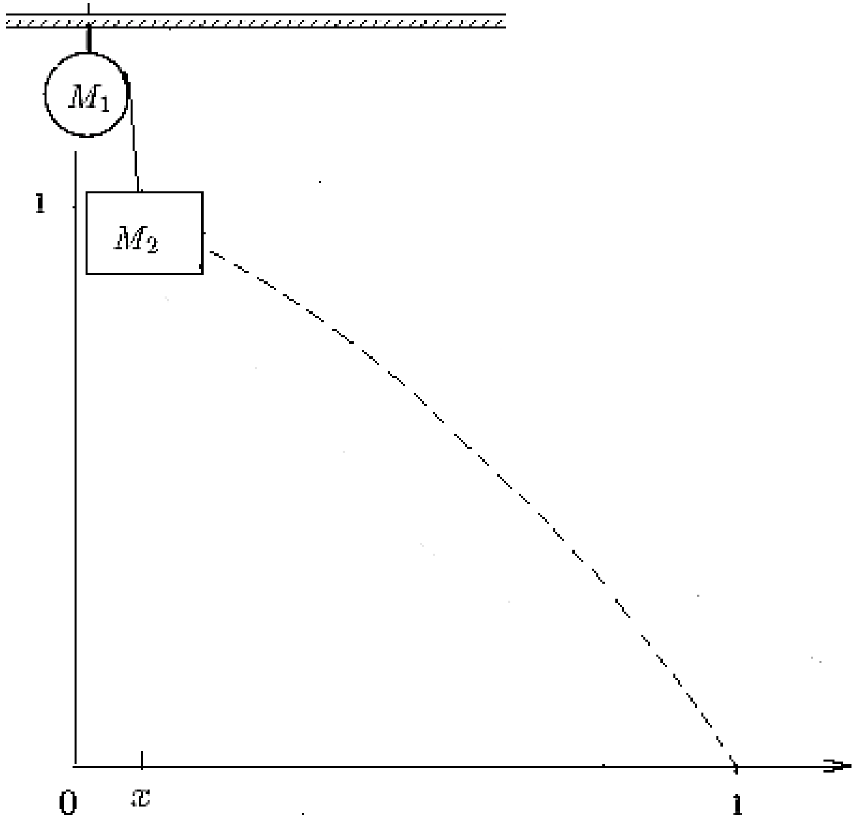


Figure 1: Control of a Crane


d
M_1
1
M_2
0
x
1



The equations of motion are given by [2] and also by Visconti [12] with unknowns $\{x, x', z, z', d, d', r, r', \theta, \tau, u_1, u_2\}$:

$$\begin{array}{rcl}
x_t - x' = 0,\ \ z_t - z' = 0,\ \ d_t - d' = 0,\ \ r_t - r' & = & 0 \\
M_{\mathit{2}}\, x'_t + \tau \sin(\theta) = 0, M_{\mathit{1}}\, d'_t + C_{\mathit{1}}\, d_t - u_1 - \tau \sin(\theta) & = & 0 \\
M_{\mathit{2}}\, z'_t + \tau \cos(\theta) - mg = 0, J\, r'_t + C_{\mathit{2}}\, r_t + C_{\mathit{3}}\, u_2 - {C_{\mathit{3}}}^2 \tau & = & 0 \\
r \sin(\theta) + d - x = 0, \quad r \cos(\theta) - z & = & 0 \\
H_1(x, z, t) = 0, \quad H_2(x, z, t) & = & 0.
\end{array}$$

The prescribed path of the mass M_2 is described by an algebraic equations $\{H_1 = 0, H_2 = 0\}$. The winch has moment of inertia J and is attached with a cable of length $r(t)$, making an angle $\theta(t)$ to the vertical.

Conclusion

We introduce a new concept: t-dominated PDE systems.

- Prolongations with respect to a single independent variable t are needed.
- No differential elimination for generic systems.
- Generalized Pryce's technique in the framework of Riquier Bases.
- Disadvantages: its limitation to square and t-dominated systems; a local method; not a universal method and does not pursue all singular cases.
- Future work: to investigate PDE models, for which our t-prolongation method promises to be practically useful.

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