Spectra of rings differentially finitely generated over a subring

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Preliminaries

Ring = an associative, a commutative ring with an identity element

- (A, Δ) is a differential ring, where $\Delta = \{\partial_1, \ldots, \partial_n\}$, $[\partial_i, \partial_j] = 0$.

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- The localization by $\{s^n\}_{n=0}^{\infty}$ is denoted by A_s.

Let $f\colon A \to B$ be a homomorphism.

- A contraction of ideal is denoted by $\mathfrak{b}^c = f^*(\mathfrak{b})$.
- An extension of ideal is denoted by $\mathfrak{a}^e$.

Relation between prime spectrum and prime Δ-spectrum

Let $f\colon A \to B$ be a Δ-homomorphism of Keigher algebras.
Let

- $f^*\colon \operatorname{Spec} B \to \operatorname{Spec} A$;
- $f^*_\Delta\colon \operatorname{Spec}^\Delta B \to \operatorname{Spec}^\Delta A$.

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Lemma (fibre lemma)

Let $\mathfrak{p} \in \operatorname{Spec}^\Delta A$. *Then* $(f^*_\Delta)^{-1}(\mathfrak{p}) \neq \emptyset \iff (f^*)^{-1}(\mathfrak{p}) \neq \emptyset$.

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Corollary (on Δ-surjectivity)

f^* *is surjective* $\Longrightarrow$ f^*_Δ *is surjective.*

Technical definitions

Let $f\colon A \to B$ be a homomorphism.

Definition

f has the going-up property (GUP), if for any

$$\mathfrak{p}_1 \subseteq \ldots \subseteq \mathfrak{p}_m \subseteq \mathfrak{p}_{m+1} \subseteq \ldots \subseteq \mathfrak{p}_n \qquad \mathfrak{p}_i \in \operatorname{Spec} f(A)$$

$$\mathfrak{q}_1 \subseteq \ldots \subseteq \mathfrak{q}_m \qquad \mathfrak{q}_i \in \operatorname{Spec} B$$

$$\mathfrak{q}_i \cap f(A) = \mathfrak{p}_i$$

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Technical definitions

Let $f: A \to B$ be a homomorphism.

Definition

f has the going-down property (GDP), if for any

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Lemma

- $(f^*)^{-1}(\mathfrak{p}) \neq \emptyset \iff (f^*_\Delta)^{-1}(\mathfrak{p}) \neq \emptyset$, *where* $\mathfrak{p} \in \operatorname{Spec}^\Delta A$.
- f^* *surjective* $\Longrightarrow f^*_\Delta$ *surjective.*
- f *has GUP* $\Longrightarrow f$ *has GUP for* Δ*-ideals.*
- f *has GDP* $\Longrightarrow f$ *has GDP for* Δ*-ideals.*

Application of pairs of properties

Let's generalize well-known "going-up" and "going-down" Cohen-Seidenberg theorems for Keigher algebras.

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Theorem ("Going-down theorem")

Let $f: A \to B$ be a Δ-homomorphism of Keigher algebras. Let also

1. *B be integral domain;*
2. *$f(A)$ be integrally closed;*
3. *B be integral over A.*

Then f has GDP for Δ-ideals.

In case of characteristic zero

Lemma

Let $A \subseteq B$ be Ritt algebras. Let

1. $B = A\{x_1, \ldots, x_n\}$ *be* Δ*-FG over* A
2. B *be integral domain*

Then $\exists\, s \in B$ *such that* $B_s = A[x_1, \ldots, x_n][y_\alpha]_{\alpha \in \Lambda}$*, where* $\{y_\alpha\}$ *are algebraic independent over* $A[x_1, \ldots, x_n]$ *and* $\operatorname{card} \Lambda \leqslant \operatorname{card} \mathbb{N}$*.*

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Theorem (main theorem)

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Corollary of main theorem

Let B be Δ-ring. Let also

- B be Δ-FG over a Δ-field $\mathbb{K}$ of characteristic zero;
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Corollary

Then $C_B = C_{\mathbb{F}}$, and $C_B / C_{\mathbb{K}}$ is algebraic field extension.

In Noetherian case

Let $f\colon A \to B$ be a Δ-homomorphism of Ritt algebras. And B is Δ-FG over A.
Suppose, additionally, that all spectra are Noetherian.
Denote $X = \operatorname{Spec}^{\Delta} A$, $Y = \operatorname{Spec}^{\Delta} B$.
Consider the properties of the map $f_{\Delta}^{*}\colon Y \to X$.

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Lemma

Let $E \subseteq Y$ be constructible. Then $f^*_{\Delta}(E)$ is constructible.

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Lemma

- *The map f has GUP for Δ-deals $=$ f^*_Δ is closed map.*
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Open maps

Theorem

Let $A \subseteq B$ be Ritt algebras. Let

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Definition

$$X := \{\mathfrak{p} \in \operatorname{Spec}^{\Delta} A \mid \exists s \in A : (A/\mathfrak{p})_s \text{ is a simple } \Delta\text{-ring}\}$$

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Lemma

The map $f^*\colon Y \to X$ is well defined.

Simple properties

Lemma

The set X is the smallest subset of $\operatorname{Spec}^{\Delta} A$ such that

- $\forall\, B$ *is Δ-FG over field $\mathbb{K}$, and*
- $\forall\, f : A \to B$ *is a Δ-homomorphism,*

we have $f^*(\operatorname{Max}^{\Delta} B) \subseteq X$.

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we have $f^*(\operatorname{Max}^{\Delta} B) \subseteq X$.

Lemma

Let $f\colon A \to B$ *be a* Δ*-homomorphism. Then*

1. f *has GUP for* Δ*-ideals* $\Longrightarrow$ f^* *is closed map from* Y *to* X*;*
2. f *has GDP for* Δ*-ideals* $\Longrightarrow$ f^* *is open map from* Y *to* X*;*
3. E *is constructible* $\Longrightarrow$ $f^*(E)$ *is constructible.*

Remark

Consider any field $\mathbb{F}$ of characteristic zero with the property:
For any algebra B such that

$$\left.\begin{array}{l} B \text{ is } \Delta\text{-FG over } \mathbb{F} \\ B \text{ is a simple } \Delta\text{-ring} \end{array}\right\} \implies B = \mathbb{F}.$$

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Universal field extension $\mathcal{U}$ of any Δ-field $\mathbb{K}$ (Kolchin).

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Let $A = \mathbb{F}\{y_1, \dots, y_n\}/\mathfrak{b}$.
The following sets coincide:

- $\{\, x \in \mathbb{F}^n \mid \forall f \in \mathfrak{b} \Rightarrow f(x) = 0 \,\}$;
- $\operatorname{Max}^\Delta A = \{\, \mathfrak{p} \in \operatorname{Spec}^\Delta A \mid A/\mathfrak{p} \text{ is a simple } \Delta\text{-ring} \,\}$;
- $\{\, \mathfrak{p} \in \operatorname{Spec}^\Delta A \mid \exists s \in A :\ (A/\mathfrak{p})_s \text{ is a simple } \Delta\text{-ring} \,\}$.

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