

O-minimality and Quantifier elimination

in some

Non Quasi-Analytic classes

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General Problem

A compact box P of \mathbb{R}^n is a **basic box** if

$$\exists (a_1, \dots, a_n) \in (\mathbb{R}_+^*)^n \quad \prod_{1 \leq i \leq n}]0, a_i[\subset P$$

• Let \mathcal{F} be a set of functions from \mathbb{R}^n to \mathbb{R} :

- which vanish outside a *basic box* P

- C^∞ on the same box P

n and P are not fixed : they depend on the functions

• Let $\overline{\mathcal{F}}$ be the closure of \mathcal{F} by

sums, products, compositions, partial derivatives and implicit functions.

We would like to understand the geometric properties of $\overline{\mathcal{F}}$. We have, among other things, two ways to explore this geometry.

The following list is not exhaustive.

- **Real Geometric** point of view :

- does $\overline{\mathcal{F}}$ satisfy a preparation (or normalization) theorem?

- does there exist a Real Nullstellensatz?

- **Model-Theoretical** point of view :

- is the complete theory of \mathbb{R} , in the language of the ordered rings, enlarged by symbols for each element of $\overline{\mathcal{F}}$, model-complete? o-minimal? Does it admit quantifier elimination?

- can we describe explicitly the axioms of this complete theory?

These two points of view are closely linked and, in the continuation, we will often pass from the one to the other.

Examples

(only polynomially bounded classes)

The following list is not exhaustive

1) Analytic and/or sub-analytic classes

- **Preparation theorem in sub-analytic classes** : obtained by van den Dries, Macintyre and Marker [vdDMM], using model-theoretical methods ; an explicit preparation theorem in these classes was obtained a few years later by Lion and Rolin [LR].
- **Real Nullstellensatz in analytic classes** : obtained by Risler [Ri].
- **O-minimality in analytic classes** (and so in sub-analytic classes) : obtained by van den Dries [vdD], based on results of Gabrielov [G].
- **Description of the complete theory of \mathbb{R} in analytic classes** (noted T_{an}) : also obtained in [vdDMM].
- **Quantifier elimination of $T_{an \cup \frac{1}{X}}$** : obtained by Denef and van den Dries [DvdD].

2) Quasi-analytic classes

- **O-minimality and model-completeness in Denjoy-Carleman classes** : obtained by Rolin, Speissegger and Wilkie [RSW].
- **Preparation theorem and quantifier elimination in general quasi-analytic classes** : obtained by Rambaud [Ra1] (condensed in [Ra2]).

Non Quasi-analytic classes

(only polynomially bounded classes)

As we see, the geometry of quasi-analytic classes is now well-known; what about non quasi-analytic classes (always in polynomially bounded cases)?

We will focus on the problem of **having preparation theorems in these classes**; then quantifier elimination follows.

We will use model-theoretical methods, ie **non-standard methods**.

Standard Definition in Non-standard models

Let K be a totally ordered field, which extends \mathbb{R} and A be a sub-ring of \mathbb{R} . The generic example for K is **Hardy fields**.

$$O = \{x \in K; (a, b) \in \mathbb{R}^2 \ a < |x| < b\}$$

is the set of **limited** elements of K over \mathbb{R} .

$$o = \{x \in K; \forall a \in \mathbb{R}_+^* \ |x| < a\}$$

is the set of **infinitesimal** elements of K over \mathbb{R} .

Moreover, we note

$$O^* = O \setminus o$$

- K has a natural increasing **valuation**, noted val .

Main property of the valuation

$$val(x) = val(y) \iff \exists \alpha \in O^* \ y = \alpha \cdot x$$

- $(x_1, \dots, x_n) \in o^n$ is **A -independent** if

$$\forall (a_1, \dots, a_n) \in A^n \ x_1^{a_1} \dots x_n^{a_n} \in O^* \iff (a_1, \dots, a_n) = 0$$

Otherwise, the tuple is **A -linked**.

- $y \in o$ is **A -linked** to $\{x_1, \dots, x_n\}$ if

$$\exists (a_1, \dots, a_n) \in A^n \ val(y) = val(x_1^{a_1} \dots x_n^{a_n})$$

From Punctual to Local properties

Let \mathcal{L} be a language and \mathcal{M} be a **saturated** \mathcal{L} -structure.
We can always place ourselves in this situation.

Let σ be a map from \mathcal{M}^n to the set of the \mathcal{L} -formulas with n free variables, such that :

$$\forall (x_1, \dots, x_n) \in \mathcal{M}^n \quad \mathcal{M} \models \sigma(x_1, \dots, x_n)[x_1, \dots, x_n]$$

then there exist F_1, \dots, F_p , p formulas of $\mathcal{I}m(\sigma)$ such that

$$\mathcal{M} \models \forall x_1 \dots \forall x_n \quad (F_1[x_1, \dots, x_n] \vee \dots \vee F_p[x_1, \dots, x_n])$$

This theorem, which directly follows from the **Theorem of Compactness** of Model-Theory, is one of the reasons to use non-standard analysis.

Roughly speaking, in saturated models, instead of studying local properties, we focus on punctual properties, ie for infinitesimal elements of these models.

In the continuation, we'll work in a **saturated elementary extension of \mathbb{R}** , noted \mathcal{M} , in the language of $\overline{\mathcal{F}}$; so \mathcal{M} is a totally ordered field which extends \mathbb{R} . We use in \mathcal{M} the notations introduced above. Therefore :

having a local preparation theorem in \mathbb{R}



having a preparation theorem
for every infinitesimal element of \mathcal{M}

Example of a Punctual Preparation Theorem

In quasi-analytic classes, we have the following theorem, which comes from the **Newton Polygon method**.

Let $(x_1, \dots, x_n) \in o_+^{*n}$ be a \mathbb{Q} -independent family and let $(t_i)_{i \in I}$ be a set of functions of $\overline{\mathcal{F}}$; there exists a \mathbb{Q} -independent family (a_1, \dots, a_n) such that :

- $x_j = a_1^{m_{1,j}} \dots a_n^{m_{n,j}}$, with $m_{k,j} \in \mathbb{N}$
- $a_j = x_1^{q_{1,j}} \dots x_n^{q_{n,j}}$, with $q_{k,j} \in \mathbb{Q}$
- for all $i \in I$

$$t_i(x_1, \dots, x_n) = a_1^{p_1} \dots a_n^{p_n} \cdot s_i(a_1, \dots, a_n)$$

with $(p_1, \dots, p_n) \in \mathbb{N}^n$ and s_i some functions of $\overline{\mathcal{F}}$, such that $\exists j \in I \quad s_j(0) \neq 0$.

Example of a desingularization

We give a simple example of a resolution of singularities for a polynomial function, using non-standard analysis. This method comes from classic processes, notably introduced by [Hensel](#).

Let us consider $P(X, Y) = -X^2 + 2.X^3 - 2.X^2.Y + X.Y^2$
and
 $(x, y) \in o_+^{*2}$ such that $P(x, y) = 0$

Factorization

$$P(x, y) = x.(-x + 2.x^2 - 2.x.y + y^2) = 0$$

Implicit Function

$Q = -X + 2.X^2 - 2.X.Y + Y^2$ satisfies the conditions to use the [theorem of implicit functions](#) around 0, ie

$$Q(0) = 0 \neq \frac{\partial Q}{\partial Y}(0)$$

Here, the implicit function φ is $\varphi(X) = X$.

Substitution

Let $y = \varphi(x) + \epsilon = x + \epsilon$

We replace y by ϵ and we obtain

$$P(x, y) = 0 \iff -x + x^2 + \epsilon^2 = 0$$

(E)-valuation

x and ϵ^2 have the same valuation

Thus, let us consider $t = \sqrt{x}$, ie $x = t^2$. So $\epsilon = \alpha.t$ with $\alpha \in O_+^*$.

As $\alpha \in O_+^*$, $\alpha = a + u$ with $a \in \mathbb{R}_+^*$ and $u \in o$.

Substitution and Factorization

We replace ϵ by t and α ; we obtain :

$$P(x, y) = 0 \iff -1 + t^2 + \alpha^2 = 0$$

Thus $a = 1$ or $a = -1$. As $\alpha > 0$, $a > 0$.

Substitution

We replace α by u ; we obtain :

$$P(x, y) = 0 \iff t^2 + 2.u + u^2 = 0$$

Implicit functions

$Q(T, U) = T^2 + 2.U + U^2$ satisfies the theorem of implicit functions.

We note ψ the implicit function.

Thus a *punctual* solution of $P(X, Y) = 0$ is

$$x = t^2 \text{ and } y = t^2 + t + t.\psi(t)$$

Therefore, on a neighbourhood U of $(0, 0)$, there exists $a \in \mathbb{R}_+^*$ such that, for all $(v, w) \in U \cap \mathbb{R}_+^* \times \mathbb{R}_+^*$

$$P(v, w) = 0 \iff \exists t \in]0, a[(v = t^2) \wedge (y = t^2 + t + t.\psi(t))$$

How to obtain a preparation theorem in certain non quasi-analytic classes ?

Very simple example. Let

$$f(X, Y) = -X^5 - X^3.Y^4 + Y^\pi.X - X^{\sqrt{26}}$$

on $\mathbb{R}_+ \times \mathbb{R}_+$. This function is not C^∞ in 0 and so f is not quasi-analytic in 0.

Let $(x, y) \in o_+^* \times o_+^*$ and $A = \mathbb{Q}(\sqrt{26}, \pi)$. Let us desingularize f in (x, y) .

- If (x, y) is A -independent, $f(x, y)$ is A -linked to $\{x, y\}$. Thus

$$f(x, y) = \theta.x^q.y^r$$

with $\theta \in O^*$ and $(q, r) \in A^2$

If we consider that an element of O^* is *almost* a function which doesn't vanish in 0 (like in Hardy fields, for example), we can say that independent families satisfy a *punctual* (or a weak) preparation theorem -even a weak normalization theorem-.

Remark : for functions C^∞ around 0, *having a weak preparation theorem in independent families* is equivalent to *being quasi-analytic*.

Thanks to theoretical-method, if f is *weakly prepared* in (x, y) , then f is *prepared* in (x, y) .

- If (x, y) is not A -independent, then we can suppose that

$$y = \alpha.x^q$$

with $\alpha \in O^*$, $q \in A$ and $q \geq 1$.

As $\alpha \in O^*$, $\alpha = a + \epsilon$ with $a \in \mathbb{R}^*$ and $\epsilon \in o$; moreover, $a > 0$ because $x > 0$ and $y > 0$.

We replace y by ϵ ; so

$$f(x, y) = x.(x^4 - x^{2+4q}.(a + \epsilon)^4 + (a + \epsilon)^\pi.x^{\pi.q} - x^{\sqrt{26}-1})$$

1) If $q > \frac{4}{\pi}$,

$$f(x, y) = x^5.(-1 - x^{4q-2}.(a + \epsilon)^4 + x^{q\pi-4}.(a + \epsilon)^\pi - x^{\sqrt{26}-5})$$

As the function

$$-1 - X^{4q-2}.(a + E)^4 + X^{q\pi-4}.(a + E)^\pi - X^{\sqrt{26}-5}$$

doesn't vanish in $(0, 0)$, f is prepared in (x, y) .

2) If $q < \frac{4}{\pi}$,

$$f(x, y) = x^{1+q\pi}.(-x^{4-q\pi} - x^{2+q.(4-\pi)}.(a + \epsilon)^4 + (a + \epsilon)^\pi - x^{\sqrt{26}-1-q\pi})$$

As the function

$$-X^{4-q\pi} - X^{2+q.(4-\pi)}.(a + E)^4 + (a + E)^\pi - X^{\sqrt{26}-1-q\pi}$$

doesn't vanish in $(0, 0)$, f is prepared in (x, y) .

3) If $q = \frac{4}{\pi}$,

$$f(x, y) = x^5 \cdot (-1 - x^{\frac{16}{\pi}-2} \cdot (a + \epsilon)^4 + (a + \epsilon)^\pi - x^{\sqrt{26}-5})$$

- If $a \neq 1$, we conclude in the same way as the above cases.

- If $a = 1$, we remark that the function $(a + E)^\pi$ admits a non trivial **development of Taylor** around 0. So

$$f(x, y) = x^5 \cdot \sum_{i \geq 1} a_i(x) \cdot \epsilon^i$$

where all the functions $a_i(X)$ are in the class $\overline{\mathcal{F}}$ generated by f and such that $\exists j \in \mathbb{N}^* \ a_j(0) \neq 0$. The functions $a_i(X)$ are not necessarily C^∞ in 0.

Now, we can use the classical methods (factorizations and implicit functions) to reduce the singularity of $\sum_{i \geq 1} a_i(X) \cdot E^i$; at the end, we obtain a punctual preparation theorem in (x, y) .

Representable Functions

The main idea in this example is that, from a *good* non quasi-analytic function, we can return to functions which admit a development of Taylor. That encourages us to define the notion of *A-representable functions*.

Let $P \subset \mathbb{R}^n$ be a basic box and $f : P \rightarrow \mathbb{R}$ be an application; f is *A-representable in 0* if :

- for all $k \in \mathbb{N}$
- for all $(a_1, \dots, a_p) \in A_+^{*p}$ ($p \in \mathbb{N}^*$)
- for all $(c_1, \dots, c_n) \in \mathbb{R}_+^{*n}$

there exist 2 open sets $U_k \subset \mathbb{R}^p$ and $V_k \subset \mathbb{R}^n$ such that $0 \in U_k$, $0 \in V_k$ and, if we note $g(X_1, \dots, X_p, Y_1, \dots, Y_n) =$

$$f(X_1^{a_1} \cdot X_2^{a_2} \dots X_p^{a_p}(c_1 + Y_1), \dots, X_1^{a_1} \cdot X_2^{a_2} \dots X_p^{a_p}(c_n + Y_n))$$

1) for all $(x_1, \dots, x_n) \in U_k \cap P$, $g(x_1, \dots, x_n, Y_1, \dots, Y_p)$ is C^∞ on V_k

2) all the partial derivatives that we obtain, are continuous on $U_k \cap P \times V_k$

Condition of Non-degeneration

We can see A -representable functions as a generalization of C^∞ functions; now, we need a generalization of the notion of quasi-analyticity. This condition is the following one.

Let f be a function which doesn't vanish identically on a basic box. f is **A -non degenerated** if for every A -independent families $(x_1, \dots, x_n) \in (o_+^*)^n$, $f(x_1, \dots, x_n)$ is A -linked to $\{x_1, \dots, x_n\}$.

Main results

Let us note $\overline{\mathcal{F}}_A$ the closure of \mathcal{F} by sums, products, compositions, factorizations, implicit functions and **powers over A** .

Let us consider also a class \mathcal{F} such that all the functions of $\overline{\mathcal{F}}_A$ are **A -representable** and **A -non degenerated**.

Theorem

In these conditions, the complete theory of \mathbb{R} in the language of $\overline{\mathcal{F}}_A$ admits quantifier elimination.

The definable functions of $\overline{\mathcal{F}}_A$ admit a preparation theorem, using only the terms of $\overline{\mathcal{F}}_A$.

Moreover we can give an explicit description of the axioms of this theory.

As a consequence, we obtain that these classes are o-minimal.

Process of desingularization

We give the idea to begin the desingularization in these classes. Let us suppose that every n -ary functions of $\overline{\mathcal{F}}_A$ are prepared in every n -tuple of strictly positive infinitesimal.

Let us prove now the preparation for $(n + 1)$ -tuples of infinitesimal.

Let $(x_1, \dots, x_n, y) \in (o_+^*)^{n+1}$ and $f \in \overline{\mathcal{F}}_A$.

- If (x_1, \dots, x_n, y) is A -independent, then

$$f(x_1, \dots, x_n, y) = \alpha \cdot x_1^{a_1} \dots x_n^{a_n} \cdot y^a$$

with $\alpha \in O^*$ and $(a_1, \dots, a_n, a) \in A^{n+1}$. Therefore f admits a weak preparation theorem in (x_1, \dots, x_n, y) .

- Otherwise, we can suppose that y is linked to (x_1, \dots, x_n) . Thus,

$$y = \beta \cdot x_1^{q_1} \dots x_n^{q_n}$$

with $\beta \in O^*$ and $(q_1, \dots, q_n) \in A^n$. Therefore,

$$\beta = b + \epsilon$$

and

$$f(x_1, \dots, x_n, y) = f(x_1, \dots, x_n, (b + \epsilon) \cdot x_1^{q_1} \dots x_n^{q_n})$$

with $b \in \mathbb{R}^*$ and $\epsilon \in o$.

As f is A -representable,

$$f(x_1, \dots, x_n, y) = \sum_{0 \leq i \leq n} c_i(x_1, \dots, x_n) \cdot \epsilon^i + \epsilon^n \cdot \delta_n$$

with $\delta_n \in o$ and $c_i \in \overline{\mathcal{F}}_A$.

Now, thanks to the hypothesis of recurrence, we can prepare all the functions c_i and then, use classical methods to reduce the singularity over ϵ .

Example of such classes

First of all, let us remark that a function C^∞ on the interior of a basic box is A -representable in 0 for every A .

So, if we consider classes such that all the functions of $\overline{\mathcal{F}}_A$ are C^∞ on the interior of a basic box, it remains only the condition of A -non degeneration to have a *good* class.

- If $A = \mathbb{Z}$, **quasi-analytic** classes satisfy \mathbb{Z} -non degeneration.
- If $A = \mathbb{R}$, \mathbb{R}_{an}^* (cf [vdDS]) satisfies \mathbb{R} -non degeneration.
- A lot of examples come from solutions of differential equations whose solutions are Gevrey or Puiseux functions.

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