Tannakian formalism for linear differential algebraic groups

Alexey Ovchinnikov

Department of Mathematics
North Carolina State University
e-mail: aiovchin@ncsu.edu

April 14, 2007

1 Motivation

- apply Tannakian formalism to find Galois groups of parametetized linear differential equations;
- express various properties of linear differential algebraic groups using representations.

2 Contents

- Representations of linear differential algebraic groups
- Neutral differential Tannakian categories
- Application to parameterized differential equations

3 Introduction

- **Differential ring** has differentiation

$$\Delta = \delta_1, \ldots, \delta_m.$$

  When $\Delta = \delta$ we say that we are in the *ordinary* case.

- For

$$\Theta = (\delta_1^{k_1}\delta_2^{k_2}\cdots\delta_n^{k_m}, k_i \geq 0)$$

  the **ring of differential polynomials** in $y_1, \ldots, y_n = Y$ is

$$\mathbf{k}[\Theta Y] =: \mathbf{k}\{y_1, \ldots, y_n\}.$$

- The ring of differential operators in δ over $\mathbf{k}$ is

$$\mathbf{k}[\delta], \quad \delta \cdot a = a \cdot \delta + \delta(a),\ a \in \mathbf{k}\,.$$

4 Linear Differential Algebraic Groups

- Let A be a finitely generated differential $\mathbf{k}$-algebra, that is
$$A = \mathbf{k}\{y_1, \ldots, y_n\}/\{f_1, \ldots, f_k\}.$$
- Linear differential algebraic group G is a **functor**
$$\{\text{differential } \mathbf{k}\text{-algebras}\} \to \{\text{groups}\},$$
$$B \mapsto \mathrm{Hom}(A, B).$$
- Representation V of G is a morphism
$$G \to \mathbf{GL}(V).$$

5 Derivatives of representations

- For $G = \mathbf{G_m} = \{(g) \mid g \in \mathbf{k}^*\}$, let $V = \operatorname{span}_{\mathbf{k}}\{v\}$ be the standard faithful representation.

  $$V^{(1)} := \mathbf{k}[\delta]_{\leqslant 1} \otimes_{\mathbf{k}} V = \operatorname{span}_{\mathbf{k}}\{v, \delta \otimes v\}.$$

  We let

  $$(g) \cdot v := gv, \quad (g) \cdot (\delta \otimes v) := \delta \otimes gv = \delta(g)v + (g \cdot \delta) \otimes v.$$

  That is,

  $$\begin{pmatrix} g \end{pmatrix} \mapsto \begin{pmatrix} g & g' \\ 0 & g \end{pmatrix}.$$

- In general, if V is a representation of G then

  $$V^{(k)} := \mathbf{k}[\delta]_{\leqslant k} \otimes_{\mathbf{k}} V, \quad g \cdot \delta^r \otimes v := \delta^r \otimes (g \cdot v).$$

6 Structure of representations

- *Differentially* generate all representations:

  Theorem 1. *Let V be a faithful representation of a linear differential algebraic group G. Then any other representation W is a subquotient of several copies of*

  $$V^{(i_1)} \otimes \ldots \otimes V^{(i_k)} \otimes V^* \otimes \ldots \otimes V^*.$$

- Distinguish groups of constant matrices via representations:

  Theorem 2. *A linear differential algebraic group $G \subset \mathbf{GL}(\mathbf{k})$ is conjugate to a group $H \subset \mathbf{GL}(C)$ iff as representations*

  $$V^{(p)} = V \oplus \bigoplus_{i=1}^{p} V_i, \quad V_i \cong V$$

  over a differential closure $\bar{\mathbf{k}}$.

7 Neutral differential Tannakian category

Definition 1. A neutral differential Tannakian category $\mathcal{C}$ over a differential field $\mathbf{k}$ is a category with $\oplus, \otimes, ^*, \underline{\mathrm{Hom}}$, and a fiber functor $\omega : \mathcal{C} \to \mathbf{Vec_k}$ and an operation

$$\partial^p : X \mapsto \partial^p X,$$

for each $p \in \mathbb{Z}_{\geqslant 1}$ commuting with ω, and respecting $\otimes$ and * via commutative diagrams, e.g.,

$$\begin{array}{ccc} \partial(X \otimes Y) & \xrightarrow{\omega} & (\omega(X) \otimes \omega(Y))^{(1)} \\ \downarrow & & \downarrow \\ \partial X \otimes Y \oplus X \otimes \partial Y & \xrightarrow{\omega} & \omega(X)^{(1)} \otimes \omega(Y) \oplus \omega(X) \otimes \omega(Y)^{(1)} \end{array}$$

8 Main theorem

Theorem 3. *For a neutral differential Tannakian category $\mathcal{C}$:*

- *the functor*

$$\mathrm{Aut}^{\otimes,\partial}(\omega) : \{\mathrm{Alg_k}(\partial)\} \to \{\mathrm{Groups}\}$$

  is representable by a differential Hopf algebra A,

- *the functor ω defines an equivalence of tensor categories*

$$\mathcal{C} \to \mathbf{Rep}_G,$$

  with G being the (pro-)linear differential algebraic group represented by A.

9 Application to parameterized linear differential equations

Linear differential equation with parameters:

$$\partial_x Y(x,t) = A(x,t)Y(x,t)$$

Differentiate it using ∂_t:

$$\partial_x \begin{pmatrix} Y \\ \partial_t Y \end{pmatrix} = \begin{pmatrix} A & 0 \\ \partial_t A0 & A \end{pmatrix} \begin{pmatrix} Y \\ \partial_t Y \end{pmatrix}$$

One continues...

- A produces a differential module (M, ∂_x)
- the prolongation of A gives a differential module $(M^{(1)}, \partial_x)$
- ...

Let K be the parameterized P.V. extension of the ∂_t, ∂_x field $\mathbf{k}$ with differentially closed $\mathbf{k}^{\partial_x}$. The fiber functor

$$\omega(N) = \ker(\partial_x, K \otimes N)$$

for each N from the category generated by M to $\mathbf{Vec}_{\mathbf{k}^{\partial_t}}$

Theorem 4. *The neutral differential Tannakian category generated by the module M is equivalent to $\mathbf{Rep}_G$, where G is the parametrized differential Galois group of K over $\mathbf{k}$.*