Jacobi's bound.
Transmission and oblivion of a mathematical notion

F. Ollivier, LIX UMRS CNRS–École polytechnique n° 7161
Mél `francois.ollivier@lix.polytechnique.fr`.
`http://www.lix.polytechnique.fr/ ollivier/indexEngl.htm`

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Introduction

In [8, 9], JACOBI has introduced a bound on the order of a system of n ordinary differential equations in n unknowns. Let $A := (a_{i,j})$ be the matrix such that $a_{i,j}$ is the order of the equation u_i in the unknown function x_j. Let $J = \max_{\sigma \in S_n} \sum_{i=1}^{m} a_{i,\sigma(i)}$. A sum $\sum_{i=1}^{m} a_{i,\sigma(i)}$ is called a *transversal sum* and J is the *maximal transversal sum*. He claims that :

JACOBI'S BOUND. — *The order of the system is bounded by J. The bound is still conjectural in the general case.*

JACOBI'S ALGORITHM. — *Jacobi gave an algorithm to compute the bound in polynomial time, instead of trying the $m!$ permutations.*

It has been forgotten and rediscovered by KUHN in 1955 ([17]), using EGERVÁRY's results (see [27] for historical details). The idea is to find a *canon*, i.e. $\lambda \in \mathbf{N}^m$ such that, in the matrix $(a_{i,j} + \lambda_i)$ one can select maximal entries in each column that are located in different rows. Jacobi's algorithm computes the unique canon with minimal λ_i. Let $\Lambda = \max_i \lambda_i$, $\alpha_i = \Lambda - \lambda_i$ and $\beta_j = \max_i a_{i,j} - \alpha_i$. The *truncated jacobian matrix* ∇ is the matrix $\left(\partial u_i / \partial x_j^{(\alpha_i + \beta_j)}\right)$.

THE TRUNCATED DETERMINANT CONDITION. — *Jacobi claims that the order of the system is equal to the bound J iff $|\nabla| \neq 0$.*

This implicitly assumes the *strong bound*, defined with the convention $\mathrm{ord}_{x_j} u_i = -\infty$ if u_i is free of x_j and its derivatives.

NORMAL FORM COMPUTATION. — *Jacobi also asserts that it is possible to compute a normal form using only λ_i derivatives of equation u_i, and that it is impossible to compute one using a smaller number of derivatives.* This implicitly assumes $|\nabla| \neq 0$; if not, a greater number of derivatives may be required. This is only genericaly true: for some particular systems, it is possible to derive u_i at most $\lambda_i - \lambda_{i+1}$ times (for $i < n$, assuming $\lambda_1 \leq \cdots \leq \lambda_n$). Jacobi also gives a bound on the order of derivation of the u_i required to compute a resolvent representation, using x_j as a primitive element—assuming it is one. Then, u_i must be derivated a number of times equal to the maximal transversal sum of the matrix obtained by suppressing the line i and the column j in A.

Unpublished manuscripts

Jacobi himself is possibly the first to have forgoten his own work. According to Koenigsberger ([13]), his manuscripts on this subject were written around 1836 and were intended to be a part of a forsaken project of long paper on differential equations. Part of this work was incorporated in his great paper on the last multiplier, but the bound himself was never published in his lifetime. The many versions of the text, containing numerous corrections, suggest that Jacobi was not satisfied of the redaction. However, these manuscripts were clearly intended for publication at the time he wrote them.

In fact these results are a by-product of his work on the *isoperimetric problem*: "Let U be a given function of the independent variable t, the dependent ones x, y, z etc. and their derivatives x', x'', etc., y', y'', etc., z', z'', etc. etc. If we propose the problem of determining the functions x, y, z in such a way that the integral

$$\int U dt$$

be *maximal or minimal* or more generaly that the differential of this integral vanish, it is known that the solution of the problem depends of the integration of the system of differential equations:

$$\begin{aligned}
0 &= \frac{\partial U}{\partial x} - \frac{d\frac{\partial U}{\partial x'}}{dt} + \frac{d^2\frac{\partial U}{\partial x''}}{dt^2} - \text{etc.},\\
0 &= \frac{\partial U}{\partial y} - \frac{d\frac{\partial U}{\partial y'}}{dt} + \frac{d^2\frac{\partial U}{\partial y''}}{dt^2} - \text{etc.},\\
0 &= \frac{\partial U}{\partial z} - \frac{d\frac{\partial U}{\partial z'}}{dt} + \frac{d^2\frac{\partial U}{\partial z''}}{dt^2} - \text{etc. etc.},
\end{aligned}$$

I will call these in the following *isoperimetric differential equations* ..." (see [10], p. 495).

If the highest order derivative of x_i in U is $x_i^{(e_i)}$, the order of x_j in the i^{th} isoperimetric equation is at most $e_i + e_j$. Then, If the e_i are not all equal to the maximal order $e := \max_i e_i$, we cannot compute a normal form without using *auxiliary equations* obtained by differentiating the i^{th} isoperimetric equation λ_i times with $\lambda_i = e - e_i$. It is also clear that $J = 2\sum_i e_i$ is equal to the order of the system, provided that the Hessian matrix $(\partial^2 U/\partial x_i \partial x_j)$ has full rank. We understand how this example can have inspired the whole theory.

In 1845, Jacobi had clearly in mind a thorough study of normal forms computation, for he wrotes: "I will expose in another paper the various ways by which this operation may be done, for this question requires many outstanding theorems that necessitate a longer exposition." (See [10].)

It is quite possible that this project was forgotten because of a change in Jacobi's life—who definitely left Koenigsberg to Berlin after a long trip in Italy—that also opened new contacts and new scientifitic issues. One may also consider a possible lack of practical examples for such a general method of computing normal forms. The algorithm may have suffer the same absence of contemporary application. Jacobi was true claiming that the problem of computing the bound was of interest by itself, but the economical questions that strongly motivated the mathematicians of the last century were not yet considered in the middle of the xixth century.

The publication of the manuscript

Jacobi's widow gave the manuscripts he left to Dirichlet who began to work for their publication with his friends Borchardt and Joachimsthal. Very few documents remain from that work and the best source seems to be Koenigsberger ([13]). It seems that the paper were in great disorder. In order to class them, they gave a number to each page. These numbers appear on the envelops were pages that seemed to form a single document or to be related were stored.

Borchardt entrusted Sigismund Cohn, who worked on the publication of some others manuscripts of Jacobi, with the documents related to the bound. He indentified (see [II/13 a)]) two sets of manuscripts suitable for publication [II/13 b), II/23 b)] and worked on a transcription [II/13 c)] of these hardly redable texts. After his death, the work was continued by Borchardt who published the first paper [8] in his journal in 1865. The second [9] was published by Clebsch in the volume *Vorlesungen über Dynamik* ([FD]) in 1866. This one was quoted by Sofya Kovalevskaya in one of her most famous papers ([16]) in 1875. The fact that these papers were written in latin did not seem to have been a trouble at that time. Cohn and Borchard could easilly write themselves some paragraphs to fill gaps in the manuscripts and Borchardt even tried to rewrite full passages in order to make them clearer. In his biography ([13]), published in 1904, Koenigsberger did not translate the many quotations in Latin, French and Italian.

Borchardt also wrote some kind of abstracts of the two papers, which show that he fully understood their content and that he did not considered what he published as devoid of rigor. A slightly ironical quotation of Jacobi himself "Tam quaestiones altioris indaginis poscuntur" (then these questions require further investigation) conclude the abstract of a part that clearly did not satisfay Borchardt and was not kept in the published version (see [II/13 b)]).

The second part of the xixth century

However, the first publication of these two papers in 1865 and 1866 and a new publication in 1890 in the 5th volume of Jacobi's complete works did not inspire further work on the subject in Germany. The works of his continuators during the xixth century were lacking both of rigor and inspiration. Nanson in 1876 ([23]), with 3 variables, and Jordan in 1883 ([12]), with 4, both tried to eliminate one variable after the other, using arguments that only work in the most general situation. Nanson does not even quote Jacobi and Jordan—who does not quote Nanson—does not have a full view of Jacobi's work. He vaguely claims that Jacobi gave an "indirect" proof and that he will give a "direct" one. The work of Chrystal in 1895 ([1]) is rigorous but he only considers the linear case with constant coefficients—Jacobi's arguments only work for all different eigenvalues. Ritt, who gives these references ([25]) also refers to a paper by Sarminski (*Communications of the University of Warsaw*, 1902) that I was unable to find.

Ritt's work

Ritt, who was known to be fluent in many languages, certainly had a better view of these two papers. However, a century after Jacobi wrote them, the style and spirit of mathematics did change. One expect rigorous proofs, but also intrinsic results, attached to geometrical objects. One thinks of varieties (or "components") and not of systems: more precisely, in [25], published in 1935, a system means a component. It is remarkable that this change also concerned a mathematician like Ritt, who also knew well some very applied style of mathematics, having worked performing computations in Naval Observatory during his studies and helped to organise a computation group working for the US artillery during World War I (see [6]).

One interest of Ritt in this subject was to secure results that could be applied to components intersection and it is not the case for Jacobi's bound (see [26]). Having developped a theory that allows to characteririze "singular components", he also wonders if the bound also stands for components that do not satisfy natural hypotheses of regularity: a difficult question that was certainly not considered by Jacobi.

The less convincing part of Jacobi's "proof" is to go from time varying systems to constant coefficients. Ritt's proof in the linear case ([25]) solves the problem. But Ritt, who only considers in [26] elimination orderings does not prove the necessary and sufficient condition for the bound to be reached, given by the non vanishing of the truncated Jacobian. Such a condition is more easily proved with an orderly ordering for an adapted order defined in this way: $\widetilde{\mathrm{ord}}_{x_j} u_i := \mathrm{ord}_{x_j} u_i - \beta_j$. His 1935 paper [25] concludes with a proof of the bound for any component of dimension zero of a system of two polynomial equations in two unknowns.

The assignment problem

In 1944, the R.A.F. tried to optimize the reaffectation of soldiers of disbanded units. No practical solution could be used before the end of World War II, but this initiated the first research on the problem. It was then considered to optimize the affectation of n workers to n tasks, $a_{i,j}$ representing the productivity of worker i if affected to task j. One looks for a maximum, with the constrain that two different workers must be given two different tasks. The MONGE problem ([22]) may be considered as a first, continuous, example of such problems (how to transport earth from one place to another minimizing the transportation price).

It is not the place to give much details about the discovery of the Hungarian method by Harold Kuhn in 1955 (see [27]). Anyway, it may be of interest to consider the situation from the standpoint of the transmission of mathematical results. Jacobi's algorithm sleeps in papers written in a dead language, with titles that cannot be related to the assignment problem. It also seems that the mathematical community was not always of a great help for the practitioner who wanted to solve his optimization problem in a short time. It is amazing that trying a finite number of possibilities may have been considered as a "mathematical solution" in the middle of the XX$^{\text{th}}$ century (see [27] p. 8). For Jacobi, trying $n!$ solution was not a solution. He claims indeed to look for *a* solution, where we understand that he is looking for an efficient one. The efficiency issue was at that time—very strongly—implicit.

One may also notice that rediscovering Jacobi's method took more than 10 years, from 1944 to 1955, and that remarkable mathematicians such as John von Neuman considered the problem. It could have been much longer if Kuhn did not translate from Hungarian Egerváry's paper [4] that allowed him to conclude. Possibly, Egerváry (1891–1958) could have contributed to the question himself, but it seems that the research concentrated in the eastern part of the world, mostly in the United States. It was also strongly motivated by economical and organizational issues and that certainly did not facilitate collaborations with eastern scientists during the cold war.

Richard COHN is the first, for the best of my knowledge, to have made a link between Jacobi's work and the assignment problem ([3]), but the news did not reach the optimization community during the 80$^{\text{ies}}$.

The second part of the XX$^{\text{th}}$ century

The second part of the XX$^{\text{th}}$ is dominated by the work of Richard COHN and his students. GREENSPAN proved a different bound, in the framework of difference algebra ([5]) in 1959. It is easily translated in differential algebra. It is in some way the dual of "Bézout's bound": $\sum_{j=1}^{n} \max_i \mathrm{ord}_{x_j} u_i$. Greenspan's bound is: $\sum_{i=1}^{n} \max_j \mathrm{ord}_{x_j} u_i$. Greenspan proved this bound for systems whose all components are of dimension zero. It was extended by COHN in 1980 ([2]) to a zero dimensional component of an arbitrary differential system.

Barbara LANDO proved in 1970 ([18]) the *"weak bound"* for order one differential systems. The *"weak bound"* means that if x_j and its derivative do not appear in u_i, we use the convention $\mathrm{ord}_{x_j} u_i = 0$. This result was translated in difference algebra

([19]) in 1972. *(Proving the strong bound for order one systems would imply the strong bound for any system.)*

There was a attempt to generalize the bound to partial differential systems ([28]), due to Tomasovic in 1976.

In 1983, Cohn proved that the bound would imply the *"dimensional conjecture": every component defined by a system of r equations has codimension at most r.*

It is quite easy to characterize the case where Jacobi's linearization argument works: it corresponds to the regularity hypothesis defined by Johnson in order to prove Janet's conjecture ([11]) in 1978: *the set of differentials $du_i^{(k)}$, $1 \leq i \leq n$, $k \in \mathbf{N}$, is linearly independent.* Under this hypothesis, Kondratieva *et al.* were able in 1982 to prove the strong bound, using first linearization as in [8], then Ritt's proof for the liearized system du. However, this result received little attention, the paper being written in Russian and difficult to find.

In 1960, Jacobi's strong bound was rediscovered independently by Volevich ([29]) for arbitrary linear systems. On may also mention the works of Magnus, [20, 21] that refer to Chrystal and Jacobi, but seems to ignore later litterature.

Beginning of the xxi$^{\text{th}}$ century

Hrushovski in 2004 ([7]) proposed a proof for the bound in difference algebra. But it does not seem possible to translate the idea to differential systems... See [24] for a proof the truncated jacobian condition in the framework of diffiety theory.

Conclusion

It is said that Jacobi once told to a student who wanted to read all the mathematical literature before starting his research: "where would you be if your father before marrying your mother had wanted to see all the girls of the world?" So, we may hope that he would have forgiven us for having forgotten some of his results.

But it could be a good idea to encourage students to read old papers and to search inspiration out of their field to preserve the unity and memory of mathematics.

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