A generalization of the Riemann-Hilbert problem

Joint work with
the late Andrey A. Bolibrukh
and with Stéphane Malek

Claude Mitschi
mitschi@math.u-strasbg.fr
IRMA
Université Louis Pasteur, Strasbourg (France)

2007 Spring Eastern Section Meeting,
Hoboken, NJ, April 14-15

Special Session in Differential Algebra

I. Classical problems for Fuchsian systems

Consider a Fuchsian differential system of order p

$$(S) \qquad \frac{dy}{dz} = B(z)y,$$

$$B(z) = \sum_{i=1}^{n} \frac{B_i}{z - a_i}, \qquad \sum_{i=1}^{n} B_i = 0.$$

and a fundamental solution Y in the neighbourhood of a non-singular point z_0.

Analytic continuation along an elementary loop γ_i (circumventing a_i counterclockwise) acts on Y by

$$Y(ze^{2i\pi}) = YG_i,$$

$G_i \in \mathrm{GL}_p(\mathbb{C})$, **monodromy matrix** w.r.t. a_i.

The matrices G_i generate the **monodromy representation** of (S)

$$\chi : \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \mathcal{D}; z_0) \longrightarrow \mathrm{GL}(p, \mathbf{C})$$

with $\mathcal{D} = \{a_1, \ldots, a_n\}$ and $\chi(\gamma_i) = G_i$ for $i = 1, \ldots, n$.

The differential **Galois group** of (S) over $\mathbb{C}(z)$ is the Zariski closure in $\mathrm{GL}(p, \mathbf{C})$ of $\mathrm{Im}(\chi)$.

Related **inverse problems:**

- Riemann-Hilbert

- inverse differential Galois

1. The classical Riemann-Hilbert problem

Historical RH problem (early note by Riemann 1850, Hilbert 21st 1900):

Given $\mathcal{D} = \{a_1, \ldots, a_n\} \subset \mathbb{P}^1_{\mathbb{C}}$, $z_0 \notin \mathcal{D}$ *and a linear representation*

$$\chi : \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \mathcal{D}; z_0) \longrightarrow \mathrm{GL}(p, \mathbf{C})$$

is there a Fuchsian (scalar) **equation**

$$y^{(p} + \alpha_1(z) y^{p-1} + \ldots + \alpha_p(z) y = 0$$

with singular locus $\mathcal{D}$ *and monodromy* χ*?*

(*Fuchsian*: each α_j has a pole of order $\leq j$ at each $a_i \in \mathcal{D}$). Additional parameters needed (apparent singularities).

The Riemann-Hilbert problem for systems: asks the same question for a Fuchsian **system**.

Answer: No, in general. (Yes, if one singularity is allowed to be regular singular only. Yes, if one G_i is diagonalizable).

Counterexamples by Bolibrukh, 1989 (mistake in Plemelj's proof, 1908, discovered by Treibich in 1979).

The modified RHP (open): characterize those χ realizable as monodromy representations.

Partial answers:

- in dimension 2 all representations χ are realizable (Dekkers, 1979, after previous work by Krylov for $n = 3$, 1956)

- in dimension 3 and 4, characterization of all realizable χ (Bolibrukh, 1990; Gladyshev, 2000).

- in arbitrary dimension $n \geq 2$, *irreducibility of* χ is sufficient (Bolibrukh and Kostov independently, 1992). Kostov estimates the codimension of the set of counterexamples in the space (monodromy groups, poles) to be $2(n-1)(p-1)$.

Variant of the RH problem (Bolibrukh, 1999): *Does there exist Fuchsian systems with given (irreducible)* χ *and given exponents* (solutions with a prescribed asymptotic behaviour at each a_i)?

Answer: No (Bolibrukh estimates codim. of the "set of counterexamples" in the moduli space of all irreducible representations).

Further variant: **RHP** for regular or Fuchsian systems on Riemann surfaces of **higher genus** (Esnault & Vieweg, 1999).

2. The inverse differential Galois problem over $\mathbb{C}(z)$

has a solution for Fuchsian systems:

Any linear algebraic group G is the differential Galois group of a Fuchsian system with coeffients in $\mathbb{C}(z)$. (Tretkoff & Tretkoff, 1979)

Proof based on Plemelj-Treibich's proof of RH and the fact that G is the Zariski closure of a finitely generated group. (One extra, apparent singularity possibly needed)

II. Local irregular systems

Consider a local meromorphic system (S)

$$z\frac{dy}{dz} = (z^{-r}\sum_{n=0}^{\infty} B_n z^n)y, \quad B_0 \neq 0,\ r \geq 0,$$

with *minimal Poincaré rank* r (in the meromorphic equivalence class of (S)).

Formal solution:

$$\hat{Y}(z) = \hat{F}(z) z^J U e^{Q(z)},$$

with $\hat{F}(z)$ a formal matrix-valued Laurent series,

$Q(z)$, U, J block-diagonal with particular, simultaneous (super-) block decomposition,

$Q(z)$ polynomial in a root of $1/z$,

J constant matrix in Jordan form.

1. Local generalized monodromy data

To (S) and $\hat{Y}$ we attach the following data:

- the **Poincaré rank** r (minimal)
- **Stokes data**:
  - *formal monodromy* $\hat{G} = U^{-1}e^{2\pi i J}U$
  - *exponential part* Q
  - *Stokes matrices*

(*and* the special block-structures of J, U, Q).

Formal monodromy: $\hat{Y}(ze^{2i\pi}) = \hat{Y}(z)\hat{G}$. *Stokes matrices*: connect convergent solutions asymptotic to $\hat{Y}$, on singular rays.

Theorem(Ramis, 1985): $\mathrm{Gal}_{\mathbb{C}(\{z\})}(S)$ *is top. generated by* $\hat{G}$*, the Stokes matrices and the exponential torus.*

.

2. Local inverse problems

"Cohomological" inverse pb (Malgrange, Sibuya): *Local generalized monodromy data are realizable as those of a local system (S).* (they define a *Stokes-sheaf* on the unit circle)

Local differential Galois inverse pb: *A linear algebraic group G is realizable as a Galois group over $\mathbb{C}(\{z\})$ iff it has a "local Galois structure", equivalently iff*

G/G^0 *is cyclic,*

$\dim(R_u/(R_u, G^0)) \leq 1$ *and*

G/G^0 *acts trivially on* $(R_u/(R_u, G^0))$.

(solved by Ramis, 1995, these conditions by M. & Singer, 1996).

Extension of a local system: Any local system (S) is analytically equivalent to one on $\mathbb{P}^1_{\mathbb{C}}$ with singularities at 0 and ∞ only, with ∞ regular singular.

Consequence: Any local Galois group is also realizable as a global Galois group by a system with two singularities, one of which regular singular.

This naturally leads to the following problem:

.

Birkhoff standard form problem (open): *Is a local system*

$$z\frac{dy}{dz} = (z^{-r}\sum_{n=0}^{\infty} B_n z^n)y, \quad B_0 \neq 0, \ r \geq 0,$$

meromorphically equivalent to a global system

$$z\frac{dy}{dz} = (\tilde{B}_r z^{-r} + \ldots + \tilde{B}_0)y?$$

Long history:

- Birkhoff 1913: original *analytic* BSF (analytic equivalence required). Counterexamples by Gantmacher and Masani 1959 (reducible). All *irreducible* systems have a solution for the analytic BSF (Bolibrukh, 1993, after previous work by Balser, Jurkat, Lutz, Peyerimhoff).

- *meromorphic* BSF : solved by Jurkat & Lutz & Peyerimhoff in dim. 2, by Balser in dim. 3, perhaps by Bolibrukh in dim. 4 (unpublished notes). Sufficient conditions were given (Turritin 1963, Balser & Bolibrukh 1997, Bolibrukh 1998).

III. The RHP for irregular systems

To any system on $\mathbb{P}^1_{\mathbb{C}}$

$$(S) \qquad \frac{dy}{dz} = B(z)y,$$

and fundam. sol. Y at a non-singular point z_0 correspond **generalized monodromy data:**

• the **singular points** $\mathcal{D} = \{a_1, \ldots, a_n\}$

• the **monodromy representation**

$$\chi : \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \mathcal{D}\,; z_0) \longrightarrow \mathrm{GL}(p, \mathbb{C})$$

• the **local generalized monodromy data** for (S) at each singular point $a \in \mathcal{D}$, including the *Stokes conditions* satisfied by the Stokes matrices at each $a \in \mathcal{D}$:

(i) $C_1 \cdot \ldots \cdot C_N \cdot \hat{G}$ is conjugated to the monodromy G at a,

(ii) $e^{Q(z)} C_i e^{-Q(z)} \asymp I$ (with Q the exponential part in $\hat{Y}$)

and the condition $\deg_z(Q) \leq r$ for the Poincaré rank.

Reduced data: if all the Poincaré ranks are minimal (in the meromorphic equivalence class of the local systems). Equivalently, if r is the least integer $\geq \deg_z Q$.

The generalized Riemann-Hilbert problem:

Is a given set $\mathcal{M}$ of reduced data realizable as the generalized monodromy data of some system (S) (hence, with minimal Poincaré rank at each singularity)?

1. If all $r_i = 0$, GRH is the **classical RH problem for Fuchsian systems**.

3. For $\mathcal{D} = \{0, \infty\}$ and $r_0 = 0$, GRH is the (meromorphic) **Birkhoff standard form problem**.

Theorem 1. *If the monodromy representation in $\mathcal{M}$ is irreducible, the GRH for $\mathcal{M}$ is solved if at least one of the prescribed singularities is without roots.*

$Without\ root$: no root of z is needed in the expression of $\hat{Y}$.

Theorem 2. *In dimension 2 and 3 the GRH problem for $\mathcal{M}$ has a solution if in the data $\mathcal{M}$ one of the singularities is without roots, and one of the singularities has at least one non-trivial Stokes matrix .*

Tentative method of solution for GRH:

1) Realize the local data (Stokes, Poincaré rank ...) by local systems $dy = \omega_i y$.

2) Glue together the ω_i to construct a canonical bundle $(\mathcal{F}, \nabla)$ with connection ∇, holomorphically trivial on $\mathbb{P}^1_{\mathbb{C}} \setminus \{a_1\}$ only, but meromorphically trivial on $\mathbb{P}^1_{\mathbb{C}}$, with the given monodromy representation.

3) Perform "admissible" (Poincaré rank-preserving) gauge transformations on the local systems to construct ∇-stable, holomorphically trivial canonical bundles.

4) Or, use unstable bundles to get solutions as well (adapting previous work of Malek for the reducible RH).

Applications to differential Galois theory

Any linear algebraic group G can be realized over $\mathbb{C}(z)$ with Fuchsian singularities. (as many, or $+1$, as topological generators of G).

What if we allow **irregular singularities**?

Notation: $L(G)$ subgp of G (topologically) generated by the $tori$ of G ($cf.$ van der Put & Singer).

π: $G \to V(G) := G/L(G)$

For any $\alpha \in V(G)$, $G_\alpha = \pi^{-1}(<\alpha>)$ has a local Galois structure.

$s(G)$:$= 1+$ minimal nb of generators of $V(G)$

G is the Galois group of a system with $s(G)$ singularities, all Fuchsian but one, possibly irregular (Ramis, 1996).

What about the Poincaré rank at the irregular singularity?

If G is *connected*, there is a bound on the Poincaré rank in terms of G (M. & Singer, 1996).

For example, if G is *connected and reductive*, it can be realized by an equation

$$\frac{dy}{dz} = (A + zB)y,$$

where A and B are constant matrices, that is, the Poincaré rank at infinity is 2.

Let $\mathcal{A} := \{\alpha \in V(G)$ belonging to a minimal set of generators of $V(G)\}$,

r_α the minimal Poincaré rank of a realization of $\pi^{-1}(<\alpha>)$ (can be computed from the group strucure of $\pi^{-1}(<\alpha>)$)

$r(G) := min_{\mathcal{A}}(r_\alpha)$.

We can apply the results of GRH and get:

Proposition 1. *Let* $r(G) = r_\alpha$ *for some* $\alpha \in \mathcal{A}$ *of a minimal set of generators* $(\alpha, \overline{M_2}, \ldots, \overline{M_{\overline{s}-1}})$ *of* $V(G)$. *Assume the family of matrices* M_i *is irreducible. Then* G *is the Galois group over* $\mathbb{C}(z)$ *of a system with no more than* $s(G)$ *singularities, all Fuchsian but one, possibly irregular, with Poincaré rank* $r(G)$.

Proposition 2. *Let G be a linear algebraic subgroup of $\mathrm{GL}(p, \mathbb{C})$, $p = 2, 3$. If*

(i) p=2, or

(ii) p=3 and for some $\alpha \in \mathcal{A}$ with $r_\alpha = r(G)$, $\pi^{-1}(< \alpha >)$ is not a formal (local) Galois group,

then G is the Galois group over $\mathbb{C}(z)$ of a differential system with no more than $s(G)$ singularities, all Fuchsian but one, irregular of Poincaré rank $r(G)$.

(*Formal Galois group*: a Galois group over the field of formal series; equivalently, if the quotient by the maximal torus is top. generated by one element)