Subfields of the complete Picard–Vessiot closure of a differential field

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$F$ a differential field of characteristic 0 and algebraically closed field of constants $C$

The Picard–Vessiot Closure $F_1$ of $F_0 = F$ is a differential extension field $F_1 \supseteq F_0$ such that

- $F_1$ is a union of Picard–Vessiot extensions of $F_0$

- Every Picard–Vessiot extension of $F_0$ embeds in $F_1$

Facts about $F_1$:

- $G(F_1/F_0)$ is proaffine and there is a Galois correspondence

- Differential automorphisms of $F_0$ [ToC] lift to $F_1$

- $F_1$ may have proper Picard–Vessiot extensions

Example $C \subset C(x) \subset C(x, \log x) \subset C(x, \log x, \text{Li}(x))$
Define inductively:

\[ F_{i+1} = (F_i)_1 \]

Then

\[ F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \]

Note inclusions can be proper: \( F_0 = C, x \in F_1 - F_0, \log x \in F_2 - F_1, \text{Li}(x) \in F_3 - F_2 \), etc [RS]

The complete Picard–Vessiot closure of \( F \) is

\[ F_\infty = \bigcup_i F_i \]

Automorphisms lift:

\[ G(F_\infty/F) \rightarrow G(F_i/F) \]

\[ G(F_\infty/F) = \varprojlim G(F_i/F) \]

\[ G(F_{i+1}/F_i) \hookrightarrow G(F_{i+1}/F) \twoheadrightarrow G(F_i/F) \]

\[ G(F_{i+1}/F_i) \text{ (pro) algebraic and} \]

\[ F_{i+1}^{G(F_{i+1}/F_i)} = F_i \text{ so } F_\infty^{G(F_\infty/F_0)} = F_0 \]
Characterization of $F_\infty$

**Theorem.** The extension $F_\infty \supseteq F$ satisfies

1. The constants of $F_\infty$ are those of $F$.

2. Every linear homogeneous differential equation over $F_\infty$ has a full set of solutions in $F_\infty$.

3. If $F_\infty \supseteq E \supseteq F$ is an intermediate differential subfield such that every linear homogeneous differential equation over $E$ has a full set of solutions in $E$ then $E = F_\infty$.

Moreover, any differential field $K \supseteq F$ with the above properties is differentiably $F$ isomorphic to $F_\infty$.

A trivial consequence of the third condition

**Corollary.** Let $E$ be a differential subfield of $F_\infty$ with $F \subseteq E$. Then $F_\infty = E_\infty$. In particular, all the fields $E_i$ can be regarded as subfields of $F_\infty$.

Consequence of this:

Automorphisms of $E$ lift to $F_\infty = E_\infty$
Reason for the corollary: $E$ a differential subfield of $F_\infty$ with $F \subseteq E$, and $L$ a monic linear differential operator over $E$. Then there is a differential subfield $E_L \subseteq F_\infty$ with $E \subseteq E_L$ such that $E_L \supseteq E$ is a Picard–Vessiot extension for $L$. 

An intermediate field $F_\infty \supseteq E \supseteq F$ is normal if $\sigma(E) = E$ all $\sigma \in G(F_\infty/F)$. 

$E \subset M \subset K$ intermediate fields, with $K$ normal. 

$G(F_\infty/E) \rightarrow G(K/E)$ defined by normality, onto by lifting 

\[
E = F_\infty^{G(F_\infty/E)} = K^{G(K/E)}
\]

so 

\[
M = K^{G(K/M)}
\]

**semi Galois Theory** 

$M \mapsto G(K/M)$ is an injection from the set of differential subfields of $K$ containing $E$ to the set of subgroups of $G(K/E)$, with right inverse $H \mapsto K^H$. 

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Iterated Picard–Vessiot (IPV) $E$: $F = E_0 \subseteq E_1 \cdots \subseteq E_n = E$ such that for each $i$ $E_{i+1}$ is a Picard–Vessiot extension of $E_i$.

$E_0, E_1, \ldots, E_n$ a defining tower for $E$.

**Theorem.** Let $E$ be a differential subfield of $F_\infty$ finitely generated over $F$. Then $E$ is contained in an iterated Picard–Vessiot extension of $F$. Conversely, if $E \supseteq F$ is a subfield of an iterated Picard–Vessiot extension then there is a differential embedding of $E$ over $F$ into $F_\infty$.

Locally Iterated Picard–Vessiot (LIPV) extension if every finite subset of $E$ belongs to an iterated Picard–Vessiot subextension of $F$ contained in $E$.

Fact: a compositum of LIPV’s in $F_\infty$ is LIPV.

**Theorem.** Let $E$ be a differential subfield of $F_\infty$. Then $E$ is contained in a locally iterated Picard–Vessiot extension of $F$. Conversely, if $E \supseteq F$ is a subfield of an locally iterated Picard–Vessiot extension then there is a differential embedding of $E$ over $F$ into $F_\infty$.
Proposition. Let $K^1$ and $K^2$ be locally iterated Picard–Vessiot extensions of $F$ inside $F_\infty$, and suppose $\tau : K^1 \to K^2$ is an $F$ differential isomorphism. Then there is an $F$ differential automorphism $\sigma$ of $F_\infty$ which restricts to $\tau$ on $K^1$.

Normality Theorem

Theorem. Let $E$ be a locally iterated Picard–Vessiot extension of $F$ contained in $F_\infty$. Then the following conditions are equivalent:

1. Every differential automorphism of $F_\infty$ over $F$ carries $E$ to itself.

2. For any no new constants extension $K$ of $F$, all differential embeddings $E \to K$ over $F$ have the same image.
Example

\[ F = C(t), \quad y \in F_1 \subseteq F_\infty, \quad y' = t^{-1} \]

\[ \{z_a \mid a \in C\} \subset F_2 \subseteq F_\infty, \quad z'_a = ((y + a)t)^{-1}. \]

\[ F\langle y \rangle = F(y) \]

\[ E_a := F\langle z_a \rangle = F(y, z_a) \]

\( F \subset F(y) \) and \( F(y) \subset E_a \) are Picard–Vessiot extensions, and \( F \subset E_a \) is an iterated Picard–Vessiot extension: a defining tower is \( F \subset F(y) \subset E_a \).

\[ \mathcal{E} = \{ E_a \mid a \in C \}. \]

The compostium \( E = F(y, \{ z_a \mid a \in C \}) \) of \( \mathcal{E} \) in \( F_\infty \) is LIPV

\[ \sigma \in G(F_\infty/F), \quad y^\sigma = y + b(\sigma) \text{ for some } b(\sigma) \in C. \]

\[ \tau \in G(F_\infty/F) \text{ then } b(\sigma \tau) = b(\sigma) + b(\tau). \]

\[ z^\sigma_a = z_a + b(\sigma) + c(\sigma, a) \text{ for some } c(\sigma, a) \in C \]

\[ c(\sigma \tau, a) = c(\sigma, a + b(\tau)) + c(\tau, a). \]

\[ G(E/F) = \text{Map}(C, \mathbb{G}_a) \rtimes \mathbb{G}_a \]

\( G_a = C \) acts on \( \text{Map}(C, \mathbb{G}_a) \) by \( a \cdot f(x) = f(x + a) \)

\[ G(E/F) \to \text{Map}(C, \mathbb{G}_a) \rtimes \mathbb{G}_a \text{ by } \sigma \mapsto (c(\sigma, \cdot), b(\sigma)). \]

\( \mathbb{G}_a \int_r \mathbb{G}_a \)