Subfields of the complete Picard–Vessiot closure of a differential field

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F a *differential field* of *characteristic* 0 and *algebraically closed* field of constants C

The *Picard–Vessiot Closure* F_1 of $F_0 = F$ is a differential extension field $F_1 \supseteq F_0$ such that

- F_1 is a union of Picard–Vessiot extensions of F_0
- Every Picard–Vessiot extension of F_0 embeds in F_1

Facts about F_1:

- $G(F_1/F_0)$ is proaffine and there is a Galois correspondence
- Differential automorphisms of F_0 [ToC] lift to F_1
- F_1 may have proper Picard–Vessiot extensions

Example $C \subset C(x) \subset C(x, \log x) \subset C(x, \log x, \mathrm{Li}(x))$

Define inductively:

$$F_{i+1} = (F_i)_1$$

Then

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots$$

Note inclusions can be proper: $F_0 = C$, $x \in F_1 - F_0$, $\log x \in F_2 - F_1$, $\mathrm{Li}(x) \in F_3 - F_2$, etc [RS]

The *complete Picard–Vessiot closure* of F is

$$F_\infty = \cup_i F_i$$

Automorphisms lift:

$G(F_\infty/F) \twoheadrightarrow G(F_i/F)$

$G(F_\infty/F) = \varprojlim G(F_i/F)$

$G(F_{i+1}/F_i) \hookrightarrow G(F_{i+1}/F) \twoheadrightarrow G(F_i/F)$

$G(F_{i+1}/F_i)$ (pro) algebraic and

$$F_{i+1}^{G(F_{i+1}/F_i)} = F_i \text{ so } F_\infty^{G(F_\infty/F_0)} = F_0$$

Characterization of F_∞

Theorem. *The extension $F_\infty \supseteq F$ satisfies*

1. *The constants of F_∞ are those of F.*

2. *Every every linear homogeneous differential equation over F_∞ has a full set of solutions in F_∞.*

3. *If $F_\infty \supseteq E \supseteq F$ is an intermediate differential subfield such that every linear homogeneous differential equation over E has a full set of solutions in E then $E = F_\infty$.*

Moreover, any differential field $K \supseteq F$ with the above properties is differentiably F isomorphic to F_∞.

A trivial consequence of the third condition

Corollary. *Let E be a differential subfield of F_∞ with $F \subseteq E$. Then $F_\infty = E_\infty$. In particular, all the fields E_i can be regarded as subfields of F_∞.*

Consequence of this:

Automorphisms of E lift to $F_\infty = E_\infty$

Reason for the corollary: E a differential subfield of F_∞ with $F \subseteq E$, and L a monic linear differential operator over E. Then there is a differential subfield $E_L \subseteq F_\infty$ with $E \subseteq E_L$ such that $E_L \supseteq E$ is a Picard–Vessiot extension for L.

An intermediate field $F_\infty \supseteq E \supseteq F$ is *normal* if $\sigma(E) = E$ all $\sigma \in G(F_\infty/F)$.

$E \subset M \subset K$ intermediate fields, with K normal.

$G(F_\infty/E) \to G(K/E)$ defined by normality, onto by lifting

$$E = F_\infty^{G(F_\infty/E)} = K^{G(K/E)}$$

so

$$M = K^{G(K/M)}$$

semi Galois Theory

$M \mapsto G(K/M)$ is an injection from the set of differential subfields of K containing E to the set of subgroups of $G(K/E)$, with right inverse $H \mapsto K^H$.

Iterated Picard–Vessiot (IPV) E: $F = E_0 \subseteq E_1 \cdots \subseteq E_n = E$ such that for each i E_{i+1} is a Picard–Vessiot extension of E_i.

$E_0, E_1, \ldots, E_n$ a *defining tower* for E.
Theorem. *Let E be a differential subfield of F_∞ finitely generated over F. Then E is contained in an iterated Picard–Vessiot extension of F. Conversely, if $E \supseteq F$ is a subfield of an iterated Picard–Vessiot extension then there is a differential embedding of E over F into F_∞.*

Locally Iterated Picard–Vessiot (LIPV) extension if every finite subset of E belongs to an iterated Picard–Vessiot subextension of F contained in E.

Fact: a compositum of LIPV's in F_∞ is LIPV.
Theorem. *Let E be a differential subfield of F_∞. Then E is contained in a locally iterated Picard–Vessiot extension of F. Conversely, if $E \supseteq F$ is a subfield of an locally iterated Picard–Vessiot extension then there is a differential embedding of E over F into F_∞.*

Proposition. *Let K^1 and K^2 be locally iterated Picard–Vessiot extensions of F inside F_∞, and suppose $\tau : K^1 \to K^2$ is an F differential isomorphism. Then there is an F differential automorphism σ of F_∞ which restricts to τ on K^1.*

Normality Theorem

Theorem. *Let E be a locally iterated Picard–Vessiot extension of F contained in F_∞. Then the following conditions are equivalent:*

1. *Every differential automorphism of F_∞ over F carries E to itself.*

2. *For any no new constants extension K of F, all differential embeddings $E \to K$ over F have the same image.*

Example

$F = C(t)$ $y \in F_1 \subseteq F_\infty$, $y' = t^{-1}$

$\{z_a \mid a \in C\} \subset F_2 \subseteq F_\infty$, $z_a' = ((y+a)t)^{-1}$.

$F\langle y\rangle = F(y)$

$E_a := F\langle z_a\rangle = F(y, z_a)$

$F \subset F(y)$ and $F(y) \subset E_a$ are Picard–Vessiot extensions, and $F \subset E_a$ is an iterated Picard–Vessiot extension: a defining tower is $F \subset F(y) \subset E_a$.

$\mathcal{E} = \{E_a \mid a \in C\}$.

The compostium $E = F(y, \{z_a \mid a \in C\})$ of $\mathcal{E}$ in F_∞ is LIPV

$\sigma \in G(F_\infty/F)$. $y^\sigma = y + b(\sigma)$ for some $b(\sigma) \in C$.

$\tau \in G(F_\infty/F)$ then $b(\sigma\tau) = b(\sigma) + b(\tau)$.

$z_a^\sigma = z_{a+b(\sigma)} + c(\sigma, a)$ for some $c(\sigma, a) \in C$

$c(\sigma\tau, a) = c(\sigma, a + b(\tau)) + c(\tau, a)$.

$G(E/F) = \text{Map}(C, \mathbb{G}_a) \rtimes \mathbb{G}_a$

$G_a = C$ acts on $\text{Map}(C, \mathbb{G}_a)$ by $a \cdot f(x) = f(x+a)$

$G(E/F) \to \text{Map}(C, \mathbb{G}_a) \rtimes \mathbb{G}_a$ by $\sigma \mapsto (c(\sigma, \cdot), b(\sigma))$.

$\mathbb{G}_a \int_r \mathbb{G}_a$