

Dimension of Difference Field Extensions

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- Let K be a difference field of zero characteristic with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$, that is, a field K considered together with mutually commuting injective endomorphisms $\alpha_1, \dots, \alpha_n$ of K . We also say that K is a σ -field with *translations* $\alpha_1, \dots, \alpha_n$.

- If α_i are automorphisms, the σ -field K is called *inversive*. In the last case we set $\sigma^* = \sigma \cup \{\alpha_1^{-1}, \dots, \alpha_n^{-1}\}$ and call K a σ^* -field.

If $n = 1$, a σ - (σ^* -) field is called *ordinary*; if $n > 1$, it is called *partial*.

- If K_0 is a subfield of K and $\alpha(K_0) \subseteq K_0$ for all $\alpha \in \sigma$, we say that K_0 is a difference (or σ -) subfield of K , and K is a difference (σ -) field extension of K_0 .

If K is inversive and $\alpha(K_0) \subseteq K_0$ for all $\alpha \in \sigma^*$, then K_0 is called a σ^* -subfield of K . We also say that we deal with σ - (or σ^* -) field extension K/K_0 .

- If K is a σ -field, then T will denote the free commutative semigroup generated by σ . The order of an element $\tau = \alpha_1^{k_1} \dots \alpha_n^{k_n} \in T$ ($k_i \in \mathbf{N}$) is defined as $ord \tau = \sum_{i=1}^n k_i$.

- If K is inversive, then Γ will denote the free commutative group generated by σ . The order of an element $\gamma = \alpha_1^{k_1} \dots \alpha_n^{k_n} \in \Gamma$ ($k_i \in \mathbf{Z}$) is defined as $ord \gamma = \sum_{i=1}^n |k_i|$.

- We set $T(r) = \{\tau \in T | ord \tau \leq r\}$ and $\Gamma(r) = \{\gamma \in \Gamma | ord \gamma \leq r\}$ for any $r \in \mathbf{N}$.

- If $B \subseteq K$ and K_0 is a σ -subfield of K , then the intersection of all σ -subfields of K containing K_0 and B is denoted by $K_0\langle B \rangle$. As a field, $K_0\langle B \rangle = K_0(\{\tau(b) | b \in B, \tau \in T\})$.

- If $K = K_0\langle B \rangle$, then the set B is called the set of σ -generators of K/K_0 . If $|B| < \infty$, $B = \{b_1, \dots, b_k\}$, we say that K is a finitely generated difference (σ -) field extension of K_0 and write $K = K_0\langle b_1, \dots, b_k \rangle$.

- Similarly, if K is an inversive difference (σ^* -) field, K_0 is a σ^* -subfield of K and $B \subseteq K$, then $K_0\langle B \rangle^*$ denotes the smallest σ^* -subfield of K containing K_0 and B . If $K = K_0\langle B \rangle^*$, the set B is called the set of σ^* -*generators* of K/K_0 . If $|B| < \infty$, $B = \{b_1, \dots, b_k\}$, we say that K is a finitely generated σ^* -field extension of K_0 and write $K = K_0\langle b_1, \dots, b_k \rangle^*$.

- Let U be a set of elements in some σ -field extension of K . We say that U is σ -*algebraically dependent* (σ -*algebraically independent*) over K , if the set $T(U) = \{\tau(u) \mid \tau \in T, u \in U\}$ is algebraically dependent (respectively, algebraically independent) over K .

If a set consisting of one element u is σ -algebraically dependent over K , then u is said to be σ -*algebraic* over K . If the set $\{\tau(u) \mid \tau \in T\}$ is algebraically independent over K , we say that u is σ -*transcendental* over K .

If L is a σ -field extension of K such that every element of L is σ -algebraic over K , we say that L is a σ -*algebraic field extension* of K or that the extension L/K is σ -algebraic. Otherwise, the extension L/K is said to be σ -transcendental.

- Let K be a difference (inversive difference) field with a basic set σ and L a σ - (respectively, σ^* -) field extension of K . Then L contains a set B such that B is σ -algebraically independent over K and any subset of L properly containing B is σ -algebraically dependent over K . Such a set B is called a *difference (or σ -) transcendence basis* of L over K .

Clearly, $B \subseteq L$ is a σ -transcendence basis of L over K if and only if every element of L is σ -algebraically dependent over $K\langle B \rangle$.

- Furthermore, all σ -transcendence bases of L over K either contain the same finite number of elements or are infinite. The *σ -transcendence degree* of L over K (denoted by $\sigma\text{-trdeg}_K L$) is the number of elements of any σ -transcendence basis of L over K , if this number is finite, or infinity in the contrary case.

Theorem 1 (1978). *Let K be a difference field with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$ and $L = K\langle \eta_1, \dots, \eta_s \rangle$ a σ -field extension of K generated by a finite family $\eta = \{\eta_1, \dots, \eta_s\}$. Then there exists a polynomial $\phi_{\eta|K}(t) \in \mathbf{Q}[t]$ with the following properties.*

$$(i) \quad \phi_{\eta|K}(r) = \text{trdeg}_K K\left(\bigcup_{i=1}^s T(r)\eta_i\right)$$

for all sufficiently large $r \in \mathbf{N}$.

(ii) $\deg \phi_{\eta|K}(t) \leq n$ and $\phi_{\eta|K}(t)$ can be represented as

$$\phi_{\eta|K}(t) = \sum_{i=0}^n a_i \binom{t+i}{i}$$

where $a_0, \dots, a_n \in \mathbf{Z}$.

(iii) The integers $a_n, d = \deg \phi_{\eta|K}(t)$ and a_d do not depend on the system of σ -generators η . Furthermore, $a_n = \sigma\text{-trdeg}_K L$.

• The polynomial $\phi_{\eta|K}(t)$ is called the *difference dimension polynomial* of the extension L/K associated with the system of σ -generators η .

• The integers $d = \deg \phi_{\eta|K}(t)$ and a_d are called the *difference* (or σ -) *type* and *typical difference* (or σ -) *transcendence degree* of L/K ; they are denoted by $\sigma\text{-type}_K L$ and $\sigma\text{-t.trdeg}_K L$, respectively.

Theorem 2. (1980). *Let K be an inversive difference field with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$ and let $L = K\langle \eta_1, \dots, \eta_s \rangle^*$ be a σ^* -field extension of K generated by a finite family $\eta = \{\eta_1, \dots, \eta_s\}$. Then there exists a polynomial $\psi_{\eta|K}(t) \in \mathbf{Q}[t]$ such that*

(i) $\psi_{\eta|K}(r) = \text{trdeg}_K K\left(\bigcup_{i=1}^s \Gamma(r)\eta_i\right)$ for all sufficiently large $r \in \mathbf{N}$.

(ii) $\deg \psi_{\eta|K}(t) \leq n$ and the polynomial $\psi_{\eta|K}(t)$ can be represented as

$$\psi_{\eta|K}(t) = \sum_{i=0}^n 2^i a_i \binom{t+i}{i}$$

where $a_0, \dots, a_n \in \mathbf{Z}$.

(iii) The integers $a_n, d = \deg \phi_{\eta|K}(t)$ and a_d do not depend on the system of σ -generators η . Furthermore, $a_n = \sigma\text{-trdeg}_K L$.

- The polynomial $\psi_{\eta|K}(t)$ is called the σ^* -*dimension polynomial* of L/K associated with the system of σ^* -generators η . The integers $d = \deg \psi_{\eta|K}$ and a_d are called the σ^* -*type* and *typical σ^* -dimension* of L/K . They are denoted by σ^* -*type* $_K L$ and σ^* -*t.trdeg* $_K L$, respectively. (If $d = n$, then σ^* -*t.trdeg* $_K L = \sigma$ -*trdeg* $_K L$.)

- We say that two finite sets of automorphisms $\sigma = \{\alpha_1, \dots, \alpha_n\}$ and $\sigma_1 = \{\tau_1, \dots, \tau_n\}$ of a field K are *equivalent* and write $\sigma \sim \sigma_1$ if there is a matrix $(k_{ij})_{1 \leq i, j \leq n} \in GL(n, \mathbf{Z})$ such that $\alpha_i = \tau_1^{k_{i1}} \dots \tau_n^{k_{in}}$ ($1 \leq i \leq n$). Clearly, in this case L/K is a σ^* -field extension if and only if it is a σ_1^* -field extension and σ -*trdeg* $_K L = \sigma_1$ -*trdeg* $_K L$.

- Considering the corresponding theory of extensions of differential fields one can expect that if L/K is a finitely generated σ^* -field extension with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$, and σ -*trdeg* $_K L = 0$, then there is a set of automorphisms $\sigma_1 = \{\beta_1, \dots, \beta_n\} \sim \sigma$ such that L is a finitely generated σ_2^* -field extension of K if L and K are treated as inversive difference fields with the basic set $\sigma_2 = \{\beta_1, \dots, \beta_{n-1}\}^*$.

The following example shows that this is not so.

Example 1. Consider the field of real numbers \mathbf{R} as a σ^* -field where $\sigma = \{\alpha = id_{\mathbf{R}}\}$. Let A be the ring of all functions $f : \mathbf{R} \rightarrow \mathbf{R}$ defined for all $x \in \mathbf{R}$ except for possibly finitely many points. Then A can be treated as a σ^* -overring of \mathbf{R} such that $\alpha f(x) = f(x + 1)$ for every $f(x) \in A$. Let η denote the function $2^{2^x} \in A$ and let $L = \mathbf{R}\langle\eta\rangle^* \subseteq A$.

Since $\alpha(\eta) = 2^{2^{x+1}} = \eta^2$, $L = \mathbf{R}(\eta, \sqrt{\eta}, \sqrt[4]{\eta}, \dots)$ where $\sqrt{\eta} = 2^{2^{x-1}} = \alpha^{-1}(\eta)$, $\sqrt[4]{\eta} = 2^{2^{x-2}} = \alpha^{-2}(\eta), \dots$

With the notation of Theorem 2,

$$\psi_{\eta|\mathbf{R}}(r) = \text{trdeg}_{\mathbf{R}}\mathbf{R}(\eta, \sqrt{\eta}, \dots, \sqrt[2^r]{\eta}) = 1$$

for every $r \in \mathbf{N}$, so $\psi_{\eta|\mathbf{R}}(t) = 1$.

Therefore, $\sigma^*\text{-trdeg}_{\mathbf{R}}L = 0$ but the field extension L/\mathbf{R} is not finitely generated (with respect to the empty basic set). Indeed, $L \neq \mathbf{R}(\eta, \sqrt{\eta}, \dots, \sqrt[2^s]{\eta}) = \mathbf{R}(\sqrt[2^s]{\eta})$ for any $s \in \mathbf{N}$, since $\sqrt[2^{s+1}]{\eta} \notin \mathbf{R}(\sqrt[2^s]{\eta})$.

(Clearly, the only sets of automorphisms that are equivalent to a basic set $\sigma = \{\alpha\}$ of an ordinary inverse difference field are σ and $\{\alpha^{-1}\}$.)

Thus, there is no direct difference analog of the corresponding result of differential algebra. However, one can obtain the following result.

Theorem 3. *Let K be an inversive difference field with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$, let L be a finitely generated σ^* -field extension of K , and let $d = \sigma^*$ -type $_K L$. Then there exists a set $\sigma_1 = \{\beta_1, \dots, \beta_n\}$ of mutually commuting automorphisms of L and a finite set $\zeta = \{\zeta_1, \dots, \zeta_q\}$ of elements of L such that $\sigma_1 \sim \sigma$ and if $\sigma_2 = \{\beta_1, \dots, \beta_d\}$, then L is an algebraic extension of the field $H = K \langle \zeta_1, \dots, \zeta_q \rangle_{\sigma_2}^*$. (H is a finitely generated σ_2^* -field extension of K when K is treated as a σ_2 -field.)*

- The last theorem gives one more confirmation to the fact that the study of algebraic (in the usual sense) finitely generated difference field extensions is a very important part of the theory of difference fields. The main characteristic used in this study is the *limit degree* of a difference field extension.

The concept of limit degree in the ordinary case was introduced by R. Cohn (1956) as follows.

- Let K be an ordinary difference ring with a basic set $\sigma = \{\alpha\}$ and $L = K\langle S \rangle$ a σ -field extension of K generated by a finite set S .

Let $S_k = \{\alpha^i(s) \mid s \in S, 0 \leq i \leq k\}$ and $d_k = K(S_k) : K(S_{k-1})$ for $k = 1, 2, \dots$ ($S_0 = S$). Then

$$\begin{aligned} d_k &= \alpha(K)(\alpha(S_k)) : \alpha(K)(\alpha(S_{k-1})) \\ &\geq K(S \cup \alpha(S_k)) : K(S \cup \alpha(S_{k-1})) = d_{k+1} \end{aligned}$$

for $k = 1, 2, \dots$.

Let $d(S) = \min\{d_k \mid k = 1, 2, \dots\}$ if some d_k is finite, or $d(S) = \infty$ if all d_k are infinite.

Lemma 1. With the above notation, $d(S)$ does not depend on the system of difference generators S of L/K .

- This lemma shows that $d(S)$ is a characteristic of the extension L/K . It is called the *limit degree* of L/K and denoted by $ld(L/K)$.

- If L/K is not finitely generated, its limit degree $ld(L/K)$ is defined to be the maximum of the limit degrees of all finitely generated difference subextensions of L/K , if this maximum exists, or ∞ if it does not.

The following two results, that give the main properties of the limit degree of ordinary difference field extensions, are due to R. Cohn.

Theorem 4. *Let K be an ordinary difference (σ -) field, M a σ -field extension of K and L/K a σ -field subextension of M/K . Then*

$$ld(M/K) = [ld(M/L)][ld(L/K)].$$

Theorem 5. *Let K be an ordinary difference field with a basic set σ and let L be a σ -field extension of K . Then:*

(i) *If the difference field extension L/K is finitely generated, then $ld(L/K) = 1$ if and only if $L = K(S)$ for some finite set $S \subseteq L$.*

(ii) *The following statements are equivalent:*

(a) *L/K is finitely generated, L is algebraic over K , and $ld(L/K) = 1$.*

(b) *$L : K$ is finite.*

- The concept of limit degree of ordinary field extensions plays the key role in the results on compatibility of difference field extensions. Two difference field extensions L/K and M/K of the same difference field K are called *incompatible* if they cannot be embedded into a common difference field extension of K .

Example 2 (R. Cohn).

Let us consider \mathbf{Q} as an ordinary difference field with the basic set $\sigma = \{\alpha = id_{\mathbf{Q}}\}$.

The field $\mathbf{Q}(i)$ ($i \in \mathbf{C}$, $i^2 = -1$) has two automorphisms that extend α : the identity mapping (denoted by the same letter α) and the complex conjugation $a + bi \mapsto a - bi$ (denoted by β). Then $\mathbf{Q}(i)$ can be treated as a difference field with the basic set $\{\alpha\}$, as well as a difference field with the basic set $\{\beta\}$. Denoting these two difference fields by G and H , respectively, we can consider them as σ -field extensions of \mathbf{Q} .

Let us show that the σ -field extensions G/\mathbf{Q} and H/\mathbf{Q} are incompatible. Indeed, suppose that there exists a σ -field extension E of \mathbf{Q} and σ -isomorphisms ϕ and ψ , respectively, of G/\mathbf{Q} and H/\mathbf{Q} into E/\mathbf{Q} .

Let $j = \phi(i)$, $k = \psi(i)$, and let γ denote the translation of E that extends α and β . Then $j^2 = k^2 = -1$ whence either $j = k$ or $j = -k$. Since $\gamma(j) = j$ and $\gamma(k) = -k$, in both cases we obtain that $j = -j$, that is, $j = 0$. This contradiction implies that G/\mathbf{Q} and H/\mathbf{Q} are incompatible.

- Let L be a σ -field extension of an ordinary difference (σ -) field K . The *core* L_K of L over K is defined to be the set of elements $a \in L$ algebraic and separable over K and such that $ld(K\langle a \rangle/K) = 1$.

It follows from Theorem 4 that L_K is a σ -field and $ld(L_K/K) = 1$. Example 2 and Theorem 5 show that L_K need not to be K , and if L/K is finitely generated, then $L = L_K$ if and only if $L : K < \infty$.

The following theorem shows that core plays an important role in the study of the problem of compatibility.

Theorem 6. *Let K be an ordinary difference field with a basic set σ and let L and M be two σ -field extensions of K . Then the following statements are equivalent.*

(i) *L/K and M/K are incompatible.*

(ii) *There exist finitely generated σ -field extensions L' and M' of K such that $L' \subseteq L$, $M' \subseteq M$, and L'/K and M'/K are incompatible.*

(iii) *L_K/K and M_K/K are incompatible.*

(iv) *L_K/K and M/K are incompatible.*

The next results give an alternative descriptions of the core.

- Let K be an ordinary inversive difference field of zero characteristic with a basic set $\sigma = \{\alpha\}$ and let L be a difference field extension of K such that L/K is algebraic (in the usual sense). Furthermore, let L_K denote the core of L over K .

Theorem 7 (R. Cohn). *With the above conventions, let $a \in L$. Then $a \in L_K$ if and only if $a \in K\langle\alpha(a)\rangle$.*

Theorem 8. *Let K and L be as above and let $L = K\langle S \rangle$ where S is a finite subset of L . Then*

$$L_K = \bigcap_{n=0}^{\infty} K\langle\alpha^n(S)\rangle.$$

Theorem 9. Let K and $L = K\langle S \rangle$ ($\text{Card } S < \infty$) be as in Theorem 8 and let $ld(L/K) = K(S) : K$.

Then

$$\bigcap_{n=0}^{\infty} K\langle\alpha^n(S)\rangle = K,$$

that is, $L_K = K$.

- Theorem 8 helps one to construct a counterexample to a statement saying that the core of a normal closure of an ordinary algebraic difference field extension L/K is the normal closure of L_K/K .

(Recall that the normal closure of an algebraic field extension L/K is the field N obtained by adjoining to K all zeros of all polynomials $Irr(a, K)$, $a \in L$.)

This incorrect statement was used in the original proof of the compatibility theorem (Theorem 6). The theorem itself is correct and its correct proof was recently obtained by R. Cohn (to be published in the Pacific J. Math.)

Example 3. Let $K = \mathbf{Q}(X)$ be the field of rational fractions in one indeterminate X over \mathbf{Q} . Then K can be treated as an inversive ordinary difference field with a basic set $\sigma = \{\alpha\}$ where $\alpha(f(X)) = f(X + 1)$ for any $f(X) \in K$.

Let L be a field extension of K obtained by adjoining to K elements $\sqrt[4]{X + j}$ ($j \in \mathbf{N}$).

Treating L as a σ -field extension of K with $\alpha(\sqrt[4]{X + j}) = \sqrt[4]{X + j + 1}$ ($j \in \mathbf{N}$) and applying Theorem 8 we obtain that $L_K = \bigcap_{n=0}^{\infty} K \langle \alpha^n(\sqrt[4]{X}) \rangle =$

$$\bigcap_{n=0}^{\infty} \mathbf{Q}(X, \sqrt[4]{X + n}, \sqrt[4]{X + n + 1}, \dots) = K,$$

so the normal closure of L_K over K coincides with K . On the other hand, the normal closure of L over K is the field

$$N = \mathbf{Q}(i, X, \sqrt[4]{X}, \sqrt[4]{X + 1}, \dots)$$

($i = \sqrt{-1}$). This field is a σ -overfield of K where the extension of α from L to N is defined by the condition $\alpha(i) = i$. Using Theorem 8 once again we obtain that $N_K = \bigcap_{n=0}^{\infty} \mathbf{Q}(i, X, \sqrt[4]{X + n}, \sqrt[4]{X + n + 1}, \dots) = \mathbf{Q}(i, X) \stackrel{\supseteq}{\neq} K$.

Another important application of limit degree is the proof of the following fundamental theorem on finitely generated field extension.

Theorem 10. *Let K be a difference field with a basic set σ , M a finitely generated σ -field extension of K , and L an intermediate difference field of M/K . Then the σ -field extension L/K is finitely generated.*

This theorem was proved by R. Cohn (1955) for ordinary case and by P. Evanovich (1984) in the case of partial difference fields. The Evanovich's proof is based on the properties of generalized limit degree defined inductively. This concept, however, can be introduced explicitly as follows.

Let K be a difference field with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$.

Let \preccurlyeq be a well-ordering of the free semigroup T such that

$\tau = \alpha_1^{k_1} \dots \alpha_n^{k_n} \preccurlyeq \tau' = \alpha_1^{l_1} \dots \alpha_n^{l_n}$ if and only if $(k_n, \dots, k_1) \leq_{lex} (l_n, \dots, l_1)$.

Furthermore, for any $r_1, \dots, r_n \in \mathbf{N}$, let

$$T(r_1, \dots, r_n) = \{\tau \in T \mid \tau \preccurlyeq \alpha_1^{r_1} \dots \alpha_n^{r_n}\}.$$

Let $L = K\langle S \rangle$ be a σ -field extension of K generated by a finite set S .

For any $(r_1, \dots, r_n) \in \mathbf{N}^n$, $r_1 \geq 1$, let $d(S; r_1, \dots, r_n) = K(T(r_1, \dots, r_n)(S)) : K(T(r_1 - 1, \dots, r_n)(S))$.

($d(S; r_1, \dots, r_n) \in \mathbf{N} \cup \{\infty\}$.)

Lemma 2 *With the above notation,*

$d(S; r_1, \dots, r_n) \geq d(S; r_1 + p_1, \dots, r_n + p_n)$ for every $(p_1, \dots, p_n) \in \mathbf{N}^n$.

Lemma 3. *With the notation of Lemma 2, $d(S) = \min\{d(S; r_1, \dots, r_n)\}$ does not depend on the finite system of σ -generators S of the extension L/K .*

- $d(S)$ is called the *limit degree* of L/K and denoted by $ld(L/K)$.

If L/K is not finitely generated, then the limit degree $ld(L/K)$ is defined as the maximum of limit degrees of finitely generated difference subextensions N/K ($K \subseteq N \subseteq L$) if this maximum exists, or ∞ if it does not.

- It is easy to check that our definition of the limit degree allows one to prove the multiplicative property of limit degree and Theorem 10 using the arguments of the R. Cohn's proof in the ordinary case.

Sketch of the proof of the Compatibility Theorem

We are going to base our proof on the following statements whose proof can be either found in [1] or can be obtained from the statements of [1, Chapter 7].

Proposition 1 (Babbitt, 1960). Let ordinary σ -field extensions L/K and M/K be separably algebraic and normal. Then L/K and M/K are compatible if and only if L_K/K and M_K/K are compatible.

Proposition 2. Let K be an ordinary difference (σ -) field and let L and M be σ -overfields of K . Then L/K and M/K are incompatible if and only if there exist intermediate σ -fields L' and M' of L/K and M/K , respectively, such that the σ -field extensions L'/K and M'/K are finitely generated and incompatible.

[1] Cohn, R. M. Difference Algebra. *Interscience*, New York, 1965.

Proposition 3. The compatibility theorem is equivalent to the following statement: If L/K is a σ -field extension such that $L_K = K$, then L/K is compatible with every difference field extension of K .

Proposition 4. Let L/K be an algebraic σ -field extension, $\sigma = \{\alpha\}$. Then α can be extended to the normal closure N of L/K . Furthermore, for any such an extension of α , the core N_K has the same underlying field.

Proposition 5. If L is a difference overfield of and ordinary difference field K , then the core of L over L_K coincides with L_K .

Proposition 6. Let L be a field extension of a field K , and let F_1 and F_2 be two intermediate fields of L/K such that either the extensions F_1/K and F_2/K are normal and separable or one of them is a finite Galois extension. Then F_1 and F_2 are linearly disjoint over K if and only if $F_1 \cap F_2 = K$.

Proposition 7. Let M be a field extension of a field K and let L_1 and L_2 be two intermediate fields of M/K which are linearly disjoint over K . Let ϕ be an isomorphism of K onto a field K' and let ψ_1 and ψ_2 be extensions of ϕ to isomorphisms of L_1 and L_2 , respectively, into an overfield N of K' such that $\psi_1(L_1)$ and $\psi_2(L_2)$ are linearly disjoint over K' . Then there exists a unique extension ψ of ϕ to an isomorphism of M into N whose contraction to L_i is ψ_i ($i = 1, 2$).

PROOF OF THE COMPATIBILITY THEOREM.

Because of the properties of the core (see page 14) and Proposition 2, it is sufficient to prove the theorem under the assumption that the difference field extensions L/K and M/K are finitely generated and separably algebraic. Furthermore, as at the beginning of the proof of Theorem VIII of [1, Chapter 7], one can justify that it is sufficient to prove our statement assuming that the difference field K is inversive. Finally, Proposition 5 shows that it is sufficient to prove the following result:

(*) *Let K be an ordinary inversive difference field with a basic set $\sigma = \{\alpha\}$, let L and M be finitely generated separably algebraic difference field extensions of K , and let $L_K = K$. Then the difference field extensions L/K and M/K are compatible.*

To prove the last statement, we adopt the following notation: if H is a difference field with a translation γ , then \hat{H} will denote the underlying field of H , and the difference field itself will be also denoted by (\hat{H}, γ) . Furthermore, let us denote the basic translation of L by α_L (this is an extension of α) and consider a normal closure \hat{N} of \hat{L} over \hat{K} as an underlying field of a difference overfield N of L with a translation α_N (the existence of an extension of α_L to \hat{N} is established in Proposition 4). We obtain the chain of difference field extensions $K = (\hat{K}, \alpha) \subseteq L = (\hat{L}, \alpha_L) \subseteq N = (\hat{N}, \alpha_N)$.

By Theorem 5, the core $F = N_K$ is a finite extension of K , so $F = K(a)$ for some $a \in F$. Let $p(y)$ be the minimum polynomial of a over \hat{K} . Then $p(y)$ can be also treated as an element of the ring $K\{y\}$ of difference polynomial in one difference indeterminate y over K .

Furthermore, for every $i \in \mathbf{N}$, let $p_i(y)$ denote the polynomial in $\hat{K}[y]$ obtained by applying α^i to every coefficient of $p(y)$.

Since $K \subseteq M$, $p(y) \in M\{y\}$. By the existence theorem for difference polynomials (see [1, Chapter 6, Theorem I]), there is a solution η of the difference polynomial $p(y)$ in some difference field extension $\Omega = (\hat{\Omega}, \alpha_\Omega)$ of M . Moreover, we can assume that $\hat{\Omega}/\hat{K}$ is normal (otherwise one can replace Ω by a difference field whose underlying field is the normal closure of $\hat{\Omega}$ over \hat{K}).

Let $S = \{p(y), p_1(y), p_2(y), \dots\} \subset K\{y\}$. Since \hat{F}/\hat{K} is normal and for every $i \in \mathbf{N}$, $\alpha_N^i(a)$ is a root of $p_i(y)$ contained in \hat{F} , \hat{F} is a splitting field of the family S over \hat{K} . On the other hand, for every $i \in \mathbf{N}$, $\alpha_\Omega^i(a)$ is a root of $p_i(y)$ contained in $\hat{\Omega}$. Since the extension $\hat{\Omega}/\hat{K}$ is normal, $\hat{\Omega}$ contains a splitting field \hat{G} of the family S over \hat{K} . Furthermore, since for every zero b of a polynomial $p_i(y)$, $\alpha_\Omega(b)$ is a zero of $p_{i+1}(y)$, the restriction α_G of α_Ω on \hat{G} is an isomorphism of the field \hat{G} into itself. Thus, $G = (\hat{G}, \alpha_G)$ is a difference subfield of $\Omega = (\hat{\Omega}, \alpha_\Omega)$.

It follows that there is a \hat{K} -isomorphism μ of \hat{F} onto \hat{G} (both fields are splitting fields of the same set of polynomials over \hat{K}). Then $\beta = \mu^{-1}\alpha_G\mu$ is an endomorphism of \hat{F} whose restriction on \hat{K} coincides with α . Thus, $F_\beta = (\hat{F}, \beta)$ is a difference overfield of K . The following diagram illustrates the arrangement of our fields.

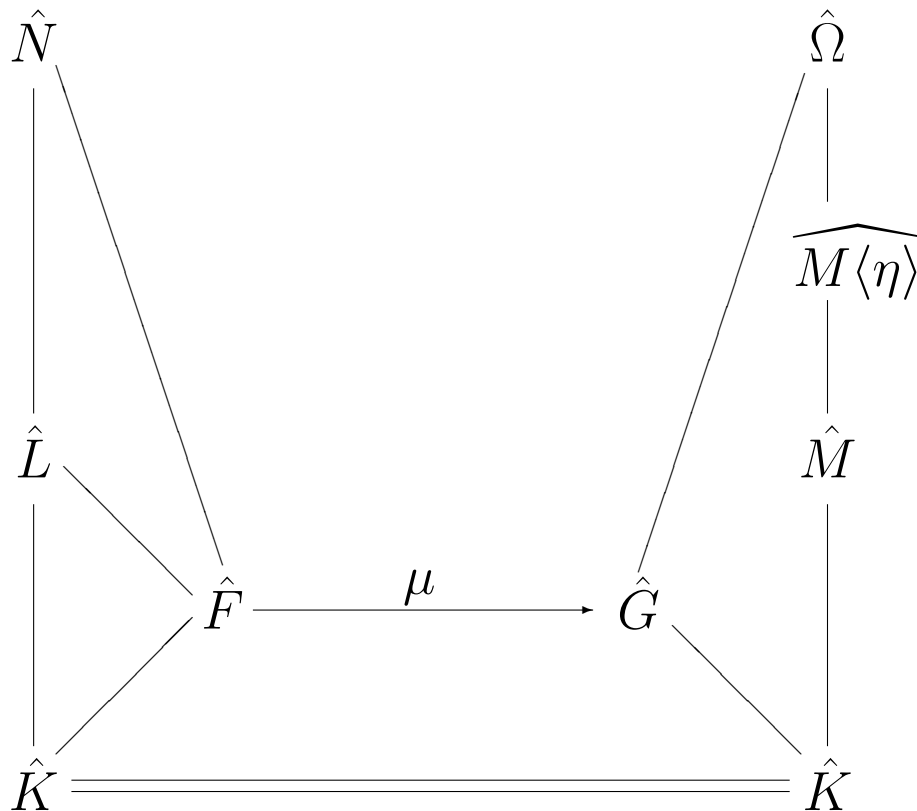


Diagram 1

Since the \hat{F} is a finite normal separable extension of \hat{K} and $\hat{F} \cap \hat{L} = \hat{L}_K = \hat{K}$, Proposition 6 shows that the field extensions \hat{F}/\hat{K} and \hat{L}/\hat{K} , as well as the extensions $\beta(\hat{F})/\hat{K}$ and $\alpha_L(\hat{L})/\hat{K}$, are linearly disjoint ($K \subseteq \alpha_L(\hat{L}) \cap \beta(\hat{F}) \subseteq \hat{L} \cap \hat{F} = K$).

Applying Proposition 7 to Diagram 2 we obtain that there exists an isomorphism $\bar{\alpha}$ from \hat{N} into itself (it is shown by a dotted line) such that the restrictions of $\bar{\alpha}$ on \hat{F} and \hat{L} coincide with β and α_L , respectively.

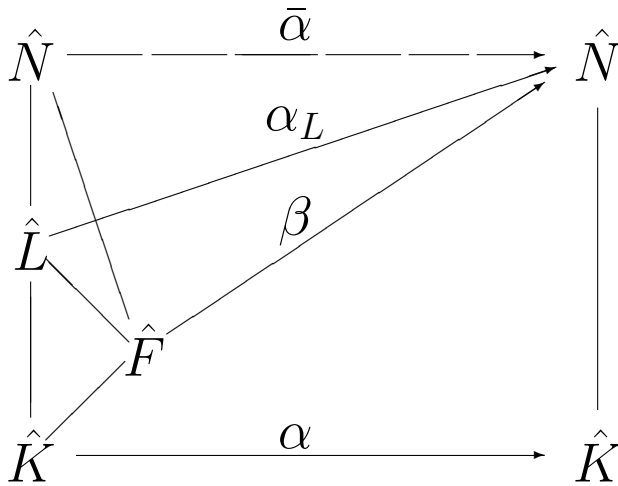


Diagram 2

Let $N' = (\hat{N}, \bar{\alpha})$. Then N' is a normal difference field extension of K with the core $F_\beta = (\hat{F}, \beta)$ (see Proposition 4).

Since μ is a difference K -isomorphism of F_β onto $G \subseteq \Omega$, the difference field extensions F_β/K and Ω_K/K are compatible. (As usual, Ω_K denotes the core of the difference field Ω over K .) Taking into account that both N'/K and Ω/K are normal, one can apply Proposition 1 and obtain that these difference field extensions are compatible. Since $L = (\hat{L}, \alpha_L)$ and M are intermediate difference fields of the extensions N'/K and Ω/K , respectively, the extensions L/K and M/K are compatible as well.