Differential central simple algebras and non-commutative Picard-Vessiot cocycles

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April 14, 2007

American Mathematical Society
Spring Eastern Section Meeting

Hoboken, NJ

Preliminaries

Notation.

K: differential field of characteristic zero with algebraically closed field of constants $\mathcal{C}$ and derivation D_K.

$(\mathcal{A}, \mathcal{D})$: differential central simple algebra (DCSA) over K, i.e., $\mathcal{A}$ is a CSA over K and $\mathcal{D}$ is a derivation of $\mathcal{A}$ extending D_K.

$[\mathcal{A}, \mathcal{D}]$: isomorphism class of $(\mathcal{A}, \mathcal{D})$ (morphisms of DCSA's over K are K-algebra homomorphisms preserving derivations).

$\mathcal{B}r(K)$: Brauer group of K.

$K_0^{\mathrm{diff}} Az(K)$: universal group on the monoid $(\{[\mathcal{A},\mathcal{D}]\},\otimes)$, where

$$[\mathcal{A}_1,\mathcal{D}_1]\otimes[\mathcal{A}_2,\mathcal{D}_2] := [A_1\otimes A_2, D_1\otimes D_2],$$

$$\mathcal{D}_1\otimes\mathcal{D}_2 := \mathcal{D}_1\otimes 1 + 1\otimes\mathcal{D}_2\,.$$

$K_0 Az(K)$: corresponding object for K when the derivations are not considered.

There is a surjection $K_0 Az(K) \to \mathcal{B}r(K)$ with kernel the subgroup $K_0\mathcal{M}at(K)$ generated by the matrix algebras.

The differential Brauer group

Consider the differential matrix algebras $(M_n(K), (\quad)')$, where $(\quad)'$ is given by D_K on matrix coordinates. Notice that

$$(M_p, (\quad)') \otimes (M_q, (\quad)') \cong (M_{pq}, (\quad)').$$

Let M^1 be the subgroup of $K_0^{\text{diff}} Az(K)$ generated by them.

The differential Brauer group is defined by:

$$\mathcal{B}r^{\text{diff}}(K) := K_0^{\text{diff}} Az(K)/M^1$$

Matrix algebras are everything here

If K is algebraically closed $\mathcal{B}r(K)$ is trivial but

$$\mathcal{B}r^{\text{diff}}(K) = M^2/M^1,$$

where M^2 is generated by the DCSA's $[M_n(K), \mathcal{D}]$ whose CSA component is a matrix algebra.

Since for any given DCSA $(A, \mathcal{D})$ there is a finite (hence uniquely differential) Galois base extension E of K such that

$$(\mathcal{A}, \mathcal{D}) \otimes E \cong (M_n(E), \overline{\mathcal{D}})$$

the interesting object here appears to be

$$M^2/M^1.$$

Definition.
Let P be an $n \times n$ matrix over K. Then $\mathcal{D}_P$ denotes the derivation of $M_n(K)$ given by

$$\mathcal{D}_P(X) = (X)' + PX - XP.$$

D_P determines P if $\mathrm{tr}(P)$ is specified. So we assume $\mathrm{tr}(P) = 0$. Also,

$$(M_n(K), \mathcal{D}_P) \otimes (M_m(K), \mathcal{D}_Q) = (M_{nm}(K), \mathcal{D}_{P \otimes I_m + I_n \otimes Q})$$

Proposition.

Let $(M_n(K), \mathcal{D})$ *be a DCSA over* K*. Then there is a unique matrix* $P \in M_n(K)$ *with* $tr(P) = 0$ *such that* $\mathcal{D} = \mathcal{D}_P$*.*

Proposition.

$(M_n(K), \mathcal{D}_P)$ is isomorphic to $(M_n(K), \mathcal{D}_Q)$ if and only if there is $H \in \mathrm{GL}_n(K)$ such that $H^{-1}H' + H^{-1}QH = P$. In particular, $(M_n(K), \mathcal{D}_P)$ is isomorphic to $(M_n(K), (\quad)')$ if and only if there is $F \in \mathrm{GL}_n(K)$ such that $F^{-1}F' = P$.

Proof.

Suppose $T : (M_n(K), \mathcal{D}_P) \to (M_n(K), D_Q)$ is a differential automorphism. As a K algebra automorphism of $M_n(K)$, T is inner: $T(X) = HXH^{-1}$, $H \in \mathrm{GL}_n(K)$. Can assume $\det(H) = 1$. Since T is differential, we have $T(\mathcal{D}_P(X)) = D_Q(T(X))$, thus $H^{-1}H' + H^{-1}QH - P$ commutes with X, hence it is central so of the form bI_n. Taking traces, we see that

$$nb = \mathrm{tr}(H^{-1}H') = \det(H)' \det(H)^{-1}.$$

Thus $b = 0$. □

Theorem.
Let $(M_n(K), \mathcal{D})$ be a differential matrix algebra over K. Then there is a PVE $E \supseteq K$ such that $(M_n(K), \mathcal{D}) \otimes_K E$ is trivial, i.e.,

$$(M_n(K), \mathcal{D}) \otimes_K E \cong (M_n(E), (\quad)').$$

Proof.
Obvious. □

Automorphisms of matrix differential algebras

Proposition.

Let $E \supset K$ be a PVE with group $G(E/K)$. The group of differential automorphisms of $(M_n(E), (\quad)')$ over K is isomorphic to $\mathrm{PGL}_n(C) \times G(E/K)$. If $A \in \mathrm{PGL}_n(C)$ and $\sigma \in G(E/K)$ the corresponding automorphism is $X \mapsto A\sigma(X)A^{-1}$.

Remember:

$$(M_n(K), \mathcal{D}) \otimes E \cong (M_n(E), (\quad)')$$

for a PVE $E \supset K$, and $\mathcal{D} = \mathcal{D}_P$ for a trace zero $P \in M_n(K)$. So

$$(M_n(K), \mathcal{D}) \otimes_K E = (M_n(E), \mathcal{D}_P).$$

Therefore,

Proposition.

Let $E \supseteq K$ be a PVE with group $G = G(E/K)$ and let $P \in M_n(K)$ and $F \in GL_n(E)$ be such that $F' = FP$. For $\sigma \in G$ let ${}_\sigma D = \sigma(F)F^{-1}$ in $GL_n(\mathcal{C})$. Then $\sigma \to (Inn_\sigma D^{-1}, \sigma)$ represents G as automorphisms so that the differential isomorphism $T_F : (M_n(E), \mathcal{D}_P) \to (M_n(E), (\quad)')$, $T_F(X) = FXF^{-1}$, is G equivariant, when G acts on the codomain via the above representation and on the domain via the action on coordinates.

Cocycles

The representation $\sigma \to (\mathrm{Inn}_\sigma D^{-1}, \sigma)$ gives a one cocycle

$$\Phi : G \to \mathrm{PGL}_n(\mathcal{C})$$
$$\sigma \mapsto T_{_\sigma D^{-1}}$$

which is in fact a coboundary since ${}_\sigma D^{-1} = F\sigma(F^{-1})$
Under the ismorphism $T_H : (M_n(K), \mathcal{D}_P) \to (M_n(K), \mathcal{D}_Q)$ the corresponding cocycle is the coboundary given by $(FH^{-1})^{-1}$.
For $H \in \mathrm{GL}_n(K)$ $\sigma(H) = H$, thus the coboundaries given by F^{-1} and $(FH^{-1})^{-1}$ are equal.

Theorem

Let $E \supseteq K$ be a PVE with differential Galois group G, and let $PGL_n(C)$ be represented as the group of inner differential automorphisms of $(M_n(E), (\quad)')$. Then there is a one to one correspondence between K isomorphism classes of matrix differential K algebras trivialized by E and homomorphisms $G \to PGL_n(C)$ which, as cocycles in $Z^1(G, PGL_n(E))$ are coboundaries.

In particular, if $(M_n(K), \mathcal{D})$ is such a K algebra, with $\mathcal{D} = \mathcal{D}_P$ and $F \in GL_n(E)$ is such that $P = F^{-1}F'$, the corresponding cocycle is $X \mapsto {}^{\sigma}X$ where ${}^{\sigma}X = {}_{\sigma}D^{-1}\sigma(X){}_{\sigma}D$ and ${}_{\sigma}D \in GL_n(C)$ is $\sigma(F)F^{-1}$; this cocycle is the coboundary associated to F^{-1}.

And if $\Lambda : G \to PGL_n(C)$ is a cocyle which is the coboundary associated to a matrix F^{-1}, for $\sigma \in G$, let $\rho(\sigma) = \Lambda(\sigma)M_n(\sigma)$ be the corresponding differential automorphism of $(M_n(E), (\quad)')$ reducing to σ on the center. Then $M_n(E)^{\rho(G)} = FM_n(K)F^{-1}$. Let $Q = F^{-1}F'$. Then $M_n(K)$ is $\mathcal{D}_Q$ stable, and

$$T_{F^{-1}} : (M_n(K), \mathcal{D}_Q) \to (M_n(E)^{\rho(G)}, (\quad)')$$

is an isomorphism, $(M_n(K), \mathcal{D}_Q)$ is trivialized by E, and Λ is the associated cocycle.

Example: $K = \mathbb{C}$

In this case $(M_n(\mathbb{C}), \mathcal{D}_p) \cong (M_n(\mathbb{C}), \mathcal{D}_Q)$ iff

$$\exists H \in \mathrm{GL}_n(\mathbb{C}) : \quad H^{-1}QH = P$$

$\therefore$ isomorphism classes of differential matrix algebras correspond to $\mathrm{GL}_n(\mathbb{C})$ orbits for the conjugation action.

We have that any two matrices whose eigenvalue sets are different correspond to different elements in $K_0^{\text{diff}}Az(K)$ and if the eigenvalues are not all zero they correspond to different elements in $\mathcal{B}r^{\text{diff}}(\mathbb{C})$.

Splitting DCSA's

Proposition.

A PVE of a finite PVE can be embedded in a PVE.

Corollary.

Let $(A, \mathcal{D})$ be a DCSA over K. Then there is a PVE E of K such that $(A, \mathcal{D}) \otimes_K E$ is a trivial differential algebra over E.