Differentiation Index of Generic DAE Systems

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ABSTRACT

The notion of the differentiation index for DAE systems of arbitrary order with generic second members is discussed. We prove an upper bound for the differentiation index in terms of the order of the input equations and we exhibit a probabilistic polynomial-time algorithm for the computation of this index. In addition, we obtain upper bounds for the regularity of the Hilbert-Kolchin function and the order of the ideal associated to the systems under consideration and we show how to compute an equivalent implicit ODE system.

1 INTRODUCTION

Different non-equivalent definitions of indices for systems of differential equations have been given (see [CG95]). Here, we are interested in the so-called *differentiation index* of an ordinary differential algebraic equation (DAE) system, which is an important invariant associated with the system (see [BCP96] and [KM06]):

- In the case of an implicit first-order zero-dimensional system, it counts the minimal number of times that all or part of the equations in the system must be differentiated in order for the derivatives of the unknowns to be determined as functions of the same unknowns. In other works, it estimates how far the given DAE system is from an equivalent explicit ODE system.
- It represents a measure of the complexity of the system from the point of view of its numerical solution: it is closely related to the condition number of the iteration matrix in the implicit Runge-Kutta method.
- It measures the solvability of the system according to the Cartan-Kuranishi theory ([Car46], [Kur57]): it counts the number of times the equations of the system must be differentiated in order to obtain all restrictions on initial conditions on the system (see also [RR94]).

The oldest algorithmic approaches to these problems for (partial) differential algebraic equations go back to the works of [Jan20], [Riq10], dealing with the construction of involutive bases (analog to the concept of Gröbner basis). A geometric alternative to this purely algebraic theory is the Cartan-Kähler theory ([Car46], [Käh34]) based on exterior systems.

2 NOTATIONS AND ASSUMPTIONS

We deal with a particular class of ordinary differential algebraic equation systems:

$$(\Sigma) := \begin{cases} f_1(X,U) & = & \dot{X}_1 \\ & \vdots & \\ f_n(X,U) & = & \dot{X}_n \\ g_1(X,U,\dot{U},\dots,U^{(e_1)}) & = & Y_1 \\ & \vdots & \\ g_r(X,U,\dot{U},\dots,U^{(e_r)}) & = & Y_r \end{cases}$$

- f_i, $g_j \in k\{X,U\}$, with k a differential field, $\mathrm{char}(k) = 0$ (for instance, $k = \mathbb{Q}(t)$ with $\dot{t} = 1$),
- $X := X_1,\dots,X_n$, $U := U_1,\dots,U_m$ differential indeterminates,
- $X^{(\ell)} := X_1^{(\ell)},\dots,X_n^{(\ell)}$, $U^{(\ell)} := U_1^{(\ell)},\dots,U_m^{(\ell)}$,
- $Y := Y_1,\dots,Y_r$, differential parameters (in this sense we say that the system is *generic*),
- e_j:= order of the highest derivative of a variable in U appearing with nonzero coefficient in g_j.

Assumption: $g_1,\dots,g_r$ are differentially algebraically independent over k as elements of $\mathrm{Frac}(k\{Y,X,U\}/[f_1-\dot{X}_1,\dots,f_n-\dot{X}_n])$.

This ensures that the differential ideal associated with the system does not contain a nonzero differential polynomial involving only variables Y.

3 MAIN RESULTS

3.1 Differentiation index via matrix rank computations

We give an algebraic definition of the index for DAE systems as (Σ) by studying stationary properties of the ranks of suitable Jacobian submatrices.

- Let $e := \max\{1, e_1, e_2, \dots, e_r\}$ be the maximum of the orders of the equations in (Σ).
- For $p \in \mathbb{N}_0$, denote $F_i^{(p)} := f_i^{(p)} - X_i^{(p+1)}$ $(1 \le i \le n)$ and $G_j^{(p)} := g_j^{(p)} - Y_j^{(p)}$ $(1 \le j \le r)$ the successive derivatives of the equations in the system,
- Consider the differential field $\mathbb{K} = k(X)\langle U\rangle$ with the derivation induced by $\dot{X}_i = f_i(X,U)$.

For each $k \in \mathbb{N}$, let $\mathfrak{J}_k$ be the Jacobian matrix of the polynomials $F, G, \dots, F^{(k-1)}, G^{(k-1)}$ with respect to the variables $X^{(e)}, U^{(e)}, \dots, X^{(e+k-1)}, U^{(e+k-1)}$.
Finally, if $(\mu_k)_{k\ge 0}$ is the integer sequence defined by

$$\mu_0 := 0 \quad \text{and} \quad \mu_k := \dim_{\mathbb{K}} \ker(\mathfrak{J}_k^t) \ \text{ for } k \ge 1,$$

we have:

Theorem: There exists $\sigma \in \mathbb{Z}_{\ge 0}$ such that:
(a) $\mu_k < \mu_{k+1}$ for every $0 \le k < \sigma$,
(b) $\mu_k = \mu_\sigma$ for every $k \ge \sigma$.

This non-negative integer σ is the differentiation index of (Σ).
Lower and upper bounds for the ranks of the matrices $\mathfrak{J}_k$ imply the following upper bound for the differentiation index in our setting:

$$\sigma \le e + n + \textstyle\sum_{j=1}^r e_j.$$

A similar bound follows from characteristic set estimates (see [Sad00]).

3.2 Differentiation index and initial conditions

The differentiation index is related to obtaining differential consequences of (Σ) with bounded order:

- Δ:= minimal (differential) ideal in $k\langle Y\rangle\{X,U\}$ containing $F_i = f_i - \dot{X}_i$, $G_j = g_j - Y_j$ and all their successive derivatives. It is a prime ideal.
- $A_\ell := k\langle Y\rangle[X,U,\dots,X^{(\ell)},U^{(\ell)}]$ for $\ell \in \mathbb{Z}_{\ge 0}$.
- $\Delta_\ell := (F,G,\dots,F^{(\ell-1)},G^{(\ell-1)})$ for $\ell \ge 1$, $\Delta_0 := (0)$.
  For every $\ell \in \mathbb{Z}_{\ge e-1}$, $\Delta_{\ell-e+1} \subset A_\ell$ and it is a prime ideal.

Theorem: Let $\sigma \in \mathbb{Z}_{\ge 0}$ be the differentiation index of the system (Σ). Then, for every $\ell \in \mathbb{Z}_{\ge e-1}$, the equality of ideals

$$\Delta_{\ell-e+1+\sigma} \cap A_\ell = \Delta \cap A_\ell$$

holds in the polynomial ring A_ℓ; furthermore, $\sigma = \min\{h \in \mathbb{Z}_{\ge 0} : \Delta_{\ell-e+1+h} \cap A_\ell = \Delta \cap A_\ell\}$. Taking $\ell = e-1$, we have

$$\sigma = \min\{h \in \mathbb{Z}_{\ge 0} : \Delta_h \cap A_{e-1} = \Delta \cap A_{e-1}\}.$$

In other words, the differentiation index of (Σ) is the minimal number of derivatives of the equations in (Σ) needed to obtain all the relations of order at most e that its solutions must verify.
This can be regarded as an alternative definition of the differentiation index of (Σ).

3.3 Computation of the differentiation index

Our approach induces a polynomial-time algorithm for the computation of the differentiation index of the system (Σ) by simple comparisons of matrix ranks. The algorithm is based on the identity $\sigma := \min\{k \in \mathbb{Z}_{\ge 0} : \mu_k = \mu_{k+1}\}$ and our upper bound on σ provides an upper bound for the maximum number of steps.

At each step, the computation of the rank of the matrix $\mathfrak{J}_k$ over the field of rational functions of a polynomial ring is required. This is done with bounded error probability by specializing the variables into randomly chosen elements of the base field.

The running time of the algorithm is polynomial in n, m, r and linear in $\log d$, L, where $d := \max\{\deg f_i, \deg g_j\}$ and L is the length of a *straight-line program* encoding f_i, g_j.

4 CONSEQUENCES

4.1 The regularity of the Hilbert-Kolchin function

The Hilbert-Kolchin function is introduced in [Kol73] in order to estimate, for each non-negative integer ℓ, the degree of freeness of the first ℓ-derivatives of the unknowns modulo the relations induced by the input equation system. More precisely, $H_\Delta : \mathbb{Z}_{\ge 0} \to \mathbb{Z}_{\ge 0}$ is defined by:

$$H_\Delta(\ell) := \mathrm{trdeg}_{k\langle Y\rangle}(\mathrm{Frac}(A_\ell/(\Delta \cap A_\ell))) \quad \text{for } \ell \in \mathbb{Z}_{\ge 0}.$$

There exists $\ell_0 \in \mathbb{Z}_{\ge 0}$ such that $H_\Delta(\ell) = (m-r)(\ell+1)+s$ for $\ell \ge \ell_0$ (see [Kol73]). The smallest ℓ_0 is the *regularity* of the function and the constant term s is called the *order of the ideal* Δ.

Theorem:

- The regularity of H_Δ is at most $e-1$.
- The order of the ideal Δ equals $e(n+r) - \mu_\sigma \le n + \sum_{j=1}^r e_j$.

It follows that the order of each element in a characteristic set of Δ with respect to an orderly ranking is bounded by e.

4.2 An equivalent implicit ODE system in dimension 0

In the case $m = r$ (that is, the differential dimension of Δ equals 0), we are able to compute an implicit ODE system which is equivalent to (Σ):

Theorem: Under the previous assumptions and notation, let $\Xi = \{\xi_1^{(\ell_1)},\dots,\xi_s^{(\ell_s)}\} \subseteq \{X,U,\dots,X^{(e-1)},U^{(e-1)}\}$ be a transcendence basis of $\mathrm{Frac}(A_{e-1}/(\Delta \cap A_{e-1}))$ over $k\langle Y\rangle$. Then:

- For $i = 1,\dots,s$ there exists a separable polynomial P_{ξ_i} such that $P_{\xi_i}(Y^{[\sigma]},\Xi,\xi_i^{(e)}) \in (F,G,\dots,F^{(\sigma)},G^{(\sigma)})$;
- For every $\eta \in \{X,U\}$ such that $\eta \ne \xi_i \ \forall i$, there exists a separable polynomial Q_η such that $Q_\eta(Y^{[\sigma-1]},\Xi,\eta^{(e-1)}) \in (F,G,\dots,F^{(\sigma-1)},G^{(\sigma-1)})$.

That is: for every $\zeta \in \{X,U\}$ there exists a separable polynomial equation for $\zeta^{(e)}$ and Ξ modulo Δ that can be obtained from the first σ derivatives of the equations in (Σ).

The polynomials P_{ξ_i} and Q_η can be interpreted as *eliminant polynomials* of algebraic geometric problems. This implies that they can be chosen with total degrees bounded by the degree of the variety $V_\sigma := \{F=0, G=0, \dots, F^{(\sigma)}=0, G^{(\sigma)}=0\}$, which is at most $d^{(\sigma+1)(n+r)}$. Moreover, they can be computed by standard algebraic elimination techniques with polynomial complexity:

Proposition: There is a probabilistic algorithm which computes implicit polynomial equations for $X^{(e)}$, $U^{(e)}$ in terms of derivatives of lower order with running time polynomial in d, n, r, $\deg(V_\sigma)$, and linear in L.

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