Iterative q difference Galois theory

Charlotte Hardouin (IWR)

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Iterative q-difference rings

Let C, be an algebraically closed field and q an element of C. Let $F = C(t)$ be the field of rational functions over C and let σ_q be the automorphism of F given by $\sigma_q(f)(t) = f(qt)$.

q-arithmetical properties

Definition 1 *Let $k \in \mathbb{Z}$. Put $[k]_q := \frac{q^k-1}{q-1}$*

1. *Let us denote by $[k]_q!$ the element of C defined by $[k]_q[k-1]_q...[1]_q$. We will say that $[k]_q!$ is the q-factorial of k.*
2. *Let us denote by $\binom{r}{k}_q$ the element of C defined by $\frac{[r]_q!}{[k]_q![(r-k)]_q!}$. We will say that $\binom{r}{k}_q$ is the q-binomial coefficient of r to k.*

Iterative q-difference ring

Definition 2 *Let R be a q-difference ring extension of F and let $\delta_R^* := (\delta_R^{(k)})_{k\in\mathbb{N}}$ be a collection of maps from R to R. The family δ_R^* is called an **iterative q-difference** of R, if all the following properties are satisfied*

1. $\delta_R^{(0)} = id$.
2. $\delta_R^{(1)} = \frac{\sigma_q - id}{(q-1)t}$
3. $\delta_R^{(k)}(x+y) = \delta_R^{(k)}(x) + \delta_R^{k}(y)$
4. $\delta^{(k)}(ab) = \sum_{i+j=k} \sigma_q^i(\delta_R^{(j)}(a))\delta_R^{(i)}(b)$.
5. $\delta_R^{(i)} \circ \delta_R^{(j)} = \binom{i+j}{i}_q \delta_R^{(i+j)}$

for all $a, b \in R$ and all $i, j, k \in \mathbb{N}$. The set of such iterative q-differences is denoted by $ID_q(R)$.
*For $\delta_R^* \in ID_q(R)$, the tuple (R, δ_R^*) is called an **iterative q-difference ring** (ID_q-ring). We say that an element c of R is a constant if $\forall k \in \mathbb{N}^*, \delta_R^{(k)}(c) = 0$. We will denote by $C(R)$ the ring of constants of R.*

Iterative q-difference modules

Now let q be a n-th primitive root of unity.

Definition 3 *Let (R, δ_R^*) be an iterative q-difference ring. Let M be a free R-module of finite type over R. We will say that (M, δ_M^*) is an iterative q-difference module if there exists a family of map $\delta_M^* = (\delta_M^{(k)})_{k\in\mathbb{N}}$, such that for all $i, j, k \in \mathbb{N}$:*

1. $\delta_M^{(0)} = id_M$.
2. $\delta_M^{(k)}$ *is an additive map from M to M.*
3. $\delta_M^{(k)}(am) = \sum_{i+j=k} \sigma_q^i(\delta_R^{(j)}(a))\delta_M^{(i)}(m)$ *for $a \in R$ and $m \in M$.*
4. $\delta_M^{(i)} \circ \delta_M^{(j)} = \binom{i+j}{i}_q \delta_M^{(i+j)}$.

The set of all iterative q-difference modules over R is denoted by $IDM_q(R)$.

Theorem 4 *Let (L, δ_L^*) be ID_q field. Then $IDM_q(L)$ is a neutral Tannakian category over L. The unit object is (L, δ_L^*).*

Iterative q-difference equation(ID_qE)

Notations 5 *Let (L, δ_L^*) be an iterative q-difference field. If,*

1. *the characteristic of the constants field C of L is zero then let us denote by $(k_C)_{k\in\mathbb{N}}$ the family $(k)_{k\in\mathbb{N}}$,*
2. *the characteristic of the constants field C of L is positive equal to p then let us denote by $(k_C)_{k\in\mathbb{N}}$ the family $\{1, (np^k)_{k\in\mathbb{N}}\}$.*

Proposition 6 *Let $M \in IDM_q(L)$ of dimension m and let $B_0 = \{b_1, ..., b_n\}$ be a basis of M. Then, there exist $\{A_k \in M_m(L)\}_{k\in\mathbb{N}}$ such that the following statements are equivalent :*

1. *For all $\mathbf{y} \in L^n$ s.t $B_0.\mathbf{y} = \sum_{i=1}^n y_i b_i \in V_M = \cap_{k\in\mathbb{N}} Ker(\delta_M^{(k)})$.*
2. *$\delta_L^{(k_C)}(\mathbf{y}) = A_k\mathbf{y}, \ \forall k \in \mathbb{N}$.*

Definition 7 *The family of equations $\{\delta_L^{(k_C)}(\mathbf{y}) = A_k\mathbf{y}\}_{k\in\mathbb{N}}$ related to the IDM_q-module (M, δ_M^*) in proposition 6 is called an **iterative q-difference equation**(ID_qE).*

Iterative q-difference Picard-Vessiot extensions

Definition 8 *Let (L, δ_L^*) be an iterative q-difference field, and let (M, δ_M^*) be an object of $IDM_q(L)$ and let $\{\delta_L^{(k_C)}(\mathbf{y}) = A_k\mathbf{y}\}_{k\in\mathbb{N}}$ be an* ***iterative q-difference equation*** *related to the IDM_q-module (M, δ_M^*), denote by $IDE_q(M)$.*
Let (R, δ_R^*) be an iterative q-difference extension of (L, δ_L^*). A matrix $Y \in Gl_n(R)$ is called a ***fundamental solution matrix*** *for $ID_qE(M)$ if $\delta_R^{(k_C)}(Y) = A_kY, \ \forall k \in \mathbb{N}$.*

The ring R is called an ***iterative q-difference Picard-vessiot ring*** *for $IDqE(M)$ (IPV_q-ring) if it fulfills the following conditions :*

1. *R is a simple IDq ring (that means that R contains no proper iterative q-difference ideal),*
2. *$IDqE(M)$ has a fundamental solution matrix Y with coefficients in R,*
3. *R is generated by the coefficients of Y and $det(Y)^{-1}$.*
4. *$C(R) = C(L)$*

Existence of Picard-Vessiot Rings and Iterative Galois groups

Theorem 9 *Let(L, δ_L^*) be an ID_q field with $C(L)$ algebraically closed and $(M, \delta_M^*) \in IDM_q(L)$ with ID_qE : $\delta_L^{(k_C)}(\mathbf{y}) = A_k\mathbf{y}$. Then, there exists an iterative q-difference Picard-Vessiot ring for the iterative q-difference equation which is unique up to iterative q-difference isomorphism.*

Definition 10 *Let (L, δ_L^*) be an iterative q-difference field, with an algebraically closed field of constants K, R/L be an iterative q-difference Picard-Vessiot ring for some iterative q-difference equation. An automorphism $\tau \in Hom_{ID_q}(R/L)$ is called an iterative difference automorphism. The group* Gal(R/L) := $\mathrm{Aut}_{\mathrm{ID_q}}$(R/L) *of all such automorphisms is called the* ***iterative q-difference Galois group*** *of the extension R/L.*

Kolchin's Approach and Galois correspondance

Proposition 11 *If R/L is an iterative q-difference Picard-Vessiot ring for some iterative q-difference equation $\delta_L^{(k_C)}(\mathbf{y}) = A_k\mathbf{y}$ with $A_k \in Gl_n(L)$ for all $k \in \mathbb{N}$, then* Gal(R/L) *is embedded in $Gl_n(C(L))$ and has a structure of reduced linear algebraic group.*

Proposition 12 *Let (L, δ_L^*) be an iterative q-difference field with field of constants C, and let M be an iterative q-difference module over L. Let us denote by R/L an IPV_q ring for M. Let $\mathcal{G} \subset Gl_n(C(L))$ be a reduced linear group such that $\mathcal{G}(C(L)) =$ Gal(R/L). Then $Spec(R)$ is a $\mathcal{G}$-torsor.*

Galois Correspondence

let (L, δ_L^*) be an ID_q-field with $C = C(L)$, $M \in ID_qL$ an ID_q-module and let E/L be an IPV_q extension for M. Let us denote by $\mathcal{G}$ a reduced linear algebraic group such that $\mathcal{G}(C) =$ Gal(E/L). Put

$$\mathfrak{H} = \{\mathcal{H} | \mathcal{H} \subset \mathcal{G} \text{is a Zariski}$$

$$\text{closed reduced linear algebraic subgroup}\},$$

and

$$\mathfrak{L} = \{T | T \text{ is an intermediate}$$

$$\text{iterative difference field } L \subset T \subset E\}.$$

Then

1. the map Ψ defined by

$$\Psi : \quad \mathfrak{H} \longrightarrow \mathfrak{L}$$

$$\mathcal{H} \longrightarrow E^{\mathcal{H}(C)}$$

is an anti-isomorphism of lattices with inverse Ψ^{-1} given by

$$\Psi^{-1}: \quad \mathfrak{L} \longrightarrow \mathfrak{H}$$

$$T \longrightarrow \mathcal{H}$$

where $\mathcal{H}(C) :=$ Gal(E/T) ;

2. if $\mathcal{H} \subset \mathcal{G}$ is a Zariski closed reduced normal subgroup, then $T := E^{\mathcal{H}(C)}$ is an iterative Picard-Vessiot extension of L with Galois group $(\mathcal{G}/\mathcal{H})(C)$.