Iterative $q$ difference Galois theory

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Special Session on Differential Algebra, II
14 April 2007
Iterative $q$-difference rings

Let $C$, be an algebraically closed field and $q$ an element of $C$. Let $F = C(t)$ be the field of rational functions over $C$ and let $\sigma_q$ be the automorphism of $F$ given by $\sigma_q(f)(t) = f(qt)$.

$q$-arithmetical properties

**Definition 1** Let $k \in \mathbb{Z}$. Put $[k]_q := \frac{q^k - 1}{q - 1}$

1. Let us denote by $[k]_q!$ the element of $C$ defined by $[k]_q[k-1]_q...[1]_q$. We will say that $[k]_q!$ is the $q$-factorial of $k$.

2. Let us denote by $\left(\begin{array}{c} r \\ k \end{array}\right)_q$ the element of $C$ defined by $\frac{[r]_q!}{[k]_q![r-k]_q!}$. We will say that $\left(\begin{array}{c} r \\ k \end{array}\right)_q$ is the $q$-binomial coefficient of $r$ to $k$. 
Iterative $q$-difference ring

**Definition 2** Let $R$ be a $q$-difference ring extension of $F$ and let $\delta_R^* := (\delta_R^{(k)})_{k \in \mathbb{N}}$ be a collection of maps from $R$ to $R$. The family $\delta_R^*$ is called an **iterative $q$-difference** of $R$, if all the following properties are satisfied

1. $\delta_R^{(0)} = \text{id}$.
2. $\delta_R^{(1)} = \frac{\sigma_q - \text{id}}{(q-1)t}$
3. $\delta_R^{(k)}(x + y) = \delta_R^{(k)}(x) + \delta_R^{k}(y)$
4. $\delta^{(k)}(ab) = \sum_{i+j=k} \sigma_q^i(\delta_R^{(j)}(a))\delta_R^{(i)}(b)$.
5. $\delta_R^{(i)} \circ \delta_R^{(j)} = \binom{i+j}{i}_q \delta_R^{(i+j)}$

for all $a, b \in R$ and all $i, j, k \in \mathbb{N}$. The set of such iterative $q$-differences is denoted by $\text{ID}_q(R)$.

For $\delta_R^* \in \text{ID}_q(R)$, the tuple $(R, \delta_R^*)$ is called an **iterative $q$-difference ring** (ID$_q$-ring). We say that an element $c$ of $R$ is a constant if $\forall k \in \mathbb{N}^*, \delta_R^{(k)}(c) = 0$. We will denote by $C(R)$ the ring of constants of $R$. 
Iterative $q$-difference modules

Now let $q$ be a $n$-th primitive root of unity.

**Definition 3** Let $(R, \delta^*_R)$ be an iterative $q$-difference ring. Let $M$ be a free $R$-module of finite type over $R$. We will say that $(M, \delta^*_M)$ is an iterative $q$-difference module if there exists a family of map $\delta^*_M = (\delta^{(k)}_M)_{k \in \mathbb{N}}$, such that for all $i, j, k \in \mathbb{N}$:

1. $\delta^{(0)}_M = id_M$.

2. $\delta^{(k)}_M$ is an additive map from $M$ to $M$.

3. $\delta^{(k)}_M (am) = \sum_{i+j=k} \sigma_q^i (\delta^{(j)}_R (a)) \delta^{(i)}_M (m)$ for $a \in R$ and $m \in M$.

4. $\delta^{(i)}_M \circ \delta^{(j)}_M = \binom{i+j}{i} q^{i+j} \delta^{(i+j)}_M$.

The set of all iterative $q$-difference modules over $R$ is denoted by $IDM_q(R)$.

**Theorem 4** Let $(L, \delta^*_L)$ be $ID_q$ field. Then $IDM_q(L)$ is a neutral Tannakian category over $L$. The unit object is $(L, \delta^*_L)$.
Iterative $q$-difference equation($ID_qE$)

Notations 5 Let $(L, \delta^*_L)$ be an iterative $q$-difference field. If,

1. the characteristic of the constants field $C$ of $L$ is zero then let us denote by $(k_C)_{k\in\mathbb{N}}$ the family $(k)_{k\in\mathbb{N}},$

2. the characteristic of the constants field $C$ of $L$ is positive equal to $p$ then let us denote by $(k_C)_{k\in\mathbb{N}}$ the family $\{1, (np^k)_{k\in\mathbb{N}}\}.$

Proposition 6 Let $M \in IDM_q(L)$ of dimension $m$ and let $B_0 = \{b_1, ..., b_n\}$ be a basis of $M.$ Then, there exist $\{A_k \in M_m(L)\}_{k\in\mathbb{N}}$ such that the following statements are equivalent:

1. For all $y \in L^n$ s.t $B_0.y = \sum_{i=1}^n y_ib_i \in V_M = \cap_{k\in\mathbb{N}} Ker(\delta_M^{(k)}).$

2. $\delta_L^{(k_C)}(y) = A_ky, \forall k \in \mathbb{N}.$

Definition 7 The family of equations $\{\delta_L^{(k_C)}(y) = A_ky\}_{k\in\mathbb{N}}$ related to the $IDM_q$-module $(M, \delta^*_M)$ in proposition 6 is called an iterative $q$-difference equation($ID_qE$).
Iterative $q$-difference Picard-Vessiot extensions

**Definition 8** Let $(L, \delta_L^*)$ be an iterative $q$-difference field, and let $(M, \delta_M^*)$ be an object of $IDM_q(L)$ and let $\{\delta_L^{(kC)}(y) = A_k y\}_{k \in \mathbb{N}}$ be an iterative $q$-difference equation related to the $IDM_q$-module $(M, \delta_M^*)$, denote by $IDE_q(M)$.

Let $(R, \delta_R^*)$ be an iterative $q$-difference extension of $(L, \delta_L^*)$. A matrix $Y \in Gl_n(R)$ is called a fundamental solution matrix for $ID_qE(M)$ if $\delta_R^{(kC)}(Y) = A_k Y$, $\forall k \in \mathbb{N}$.

The ring $R$ is called an iterative $q$-difference Picard-vessiot ring for $ID_qE(M)$ (IPV$_q$-ring) if it fulfills the following conditions:

1. $R$ is a simple $ID_q$ ring (that means that $R$ contains no proper iterative $q$-difference ideal),

2. $ID_qE(M)$ has a fundamental solution matrix $Y$ with coefficients in $R$,

3. $R$ is generated by the coefficients of $Y$ and $\text{det}(Y)^{-1}$.

4. $C(R) = C(L)$
Existence of Picard-Vessiot Rings and Iterative Galois groups

**Theorem 9** Let $(L, \delta^*_L)$ be an $ID_q$ field with $C(L)$ algebraically closed and $(M, \delta^*_M) \in IDM_q(L)$ with $ID_qE : \delta^{(kC)}_L(y) = A_k y$. Then, there exists an iterative $q$-difference Picard-Vessiot ring for the iterative $q$-difference equation which is unique up to iterative $q$-difference isomorphism.

**Definition 10** Let $(L, \delta^*_L)$ be an iterative $q$-difference field, with an algebraically closed field of constants $K$, $R/L$ be an iterative $q$-difference Picard-Vessiot ring for some iterative $q$-difference equation. An automorphism $\tau \in Hom_{ID_q}(R/L)$ is called an iterative difference automorphism. The group $\text{Gal}(R/L) := \text{Aut}_{ID_q}(R/L)$ of all such automorphisms is called the **iterative $q$-difference Galois group** of the extension $R/L$. 
Kolchin’s Approach and Galois correspondence

**Proposition 11** If $R/L$ is an iterative $q$-difference Picard-Vessiot ring for some iterative $q$-difference equation $\delta^{(kC)}_L(y) = A_ky$ with $A_k \in \text{Gl}_n(L)$ for all $k \in \mathbb{N}$, then $\text{Gal}(R/L)$ is embedded in $\text{Gl}_n(C(L))$ and has a structure of reduced linear algebraic group.

**Proposition 12** Let $(L, \delta^*_L)$ be an iterative $q$-difference field with field of constants $C$, and let $M$ be an iterative $q$-difference module over $L$. Let us denote by $R/L$ an_IPV_q ring for $M$. Let $G \subset \text{Gl}_n(C(L))$ be a reduced linear group such that $G(C(L)) = \text{Gal}(R/L)$. Then $\text{Spec}(R)$ is a $G$-torsor.
Galois Correspondence

let \((L, \delta^*_L)\) be an \(ID_q\)-field with \(C = C(L)\), \(M \in ID_qL\) an \(ID_q\)-module and let \(E/L\) be an \(IPV_q\) extension for \(M\). Let us denote by \(\mathcal{G}\) a reduced linear algebraic group such that \(\mathcal{G}(C) = \text{Gal}(E/L)\). Put

\[\mathfrak{H} = \{\mathcal{H} | \mathcal{H} \subset \mathcal{G} \text{ is a Zariski closed reduced linear algebraic subgroup}\},\]

and

\[\mathcal{L} = \{T | T \text{ is an intermediate iterative difference field } L \subset T \subset E\}.\]

Then

1. the map \(\Psi\) defined by

\[\Psi : \mathfrak{H} \longrightarrow \mathcal{L}\]

\[\mathcal{H} \rightarrow E^{\mathcal{H}(C)}\]
is an anti-isomorphism of lattices with inverse $\psi^{-1}$ given by

$$\psi^{-1} : \mathcal{L} \rightarrow \mathcal{H}$$

$$T \rightarrow \mathcal{H}$$

where $\mathcal{H}(C) := \text{Gal}(E/T)$;

2. if $\mathcal{H} \subset \mathcal{G}$ is a Zariski closed reduced normal subgroup, then $T := E^{\mathcal{H}(C)}$ is an iterative Picard-Vessiot extension of $L$ with Galois group $(\mathcal{G}/\mathcal{H})(C)$. 