Local differential Galois group and adjoint representation.

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The differential Galois group of a linear differential operator or, equivalently, a linear differential system is an algebraic group $G$ reflecting properties of the operator.

One would like to give an interpretation of the Weyl group $W$ of $G$, known to give important informations in the algebraic theory of groups, in terms of the differential operator.

Here we deal with the local situation where one can take advantage of Ramis’ description of the differential Galois group.
The local group $G$.

Let

$$Y' = A(x)Y$$  \hspace{1cm} (1)$$

be a linear differential system of dimension $n$, given by a $n \times n$ matrix $A$ with entries in the field $\mathbb{C}(\{x\})$ of meromorphic convergent power series in the complex variable $x$. The differential Galois group is

$$G = Aut_{diff}(K, \mathbb{C}(\{x\}))$$

where $K$ is any Picard-Vessiot extension of $\mathbb{C}(\{x\})$ associated to the system.
As Picard-Vessiot extension one can use the field generated by a formal fundamental matrix of solutions of the form

$$\hat{Y} = F(x)x^Le^Q(t)$$

where $F \in GL_n(\mathbb{C}((x)))$ is an invertible matrix of meromorphic formal series, $L \in gl_n(\mathbb{C})$ is a constant matrix and $Q(t)$ is a diagonal matrix of polynomials (without constant terms) in the variable $t$ where $t^\nu = x$ for some $\nu \in \mathbb{N}^*$. 
One can describe $G$ by its action on the dimension $n$ vector space $V$ of formal solutions of the system (1). The space $V$ is spanned by the columns of $\hat{Y}$ and $G$ appears as a closed subgroup of $GL_n(\mathbb{C})$. Its Lie-algebra $\mathfrak{g}$ is then a sub-Lie-algebra of $\mathfrak{gl}_n(\mathbb{C})$ on which $G$ acts (adjoint action).

Let $\mathcal{D} = \{q \mid q$ is one entry of $Q\}$ be the set of determinant factors. Then

$$V = \bigoplus_{q \in \mathcal{D}} V_q$$

where $V_q$ is the subspace of solutions the exponential part of which is $e^q$. 
Theorem 1 (Ramis). The group $G$ is topologically generated by the exponential torus, the formal monodromy and the Stokes matrices.

The formal monodromy is the element $\gamma$ of $G$ acting on $V$ by the matrix $M$ such that

$$\hat{Y}(e^{2i\pi x}) = \hat{Y}(x)M.$$ 

It permutes the $V_q : \gamma(V_q) = V_{\gamma(q)}$. 
The exponential torus $T_e$ reflects the integral relations between the determinant factors.

A $\tau \in T_e$ acts via a matrix of the form

$$\bigoplus_{q \in \mathcal{D}} \lambda_q I_{r_q}$$

where $r_q = \dim V_q$ and $\lambda_q \in \mathbb{C}^*$. Among them $\sigma = \dim \bigoplus_{q \in \mathcal{D}} q\mathbb{Z}$ are free, the others are monomials reflecting multiplicatively the additive relations among the determinant factors.

Thus $T_e$ is a torus of dimension $\sigma$. 
Let $Q = \{q_i - q_j \mid q_i, q_j \in D, q_i \neq q_j\}$.

A direction $d$ is singular if for some $q \in Q$, $e^q$ has maximal decay in direction $d$.

One then writes $d \in Fr(q)$.

Modulo $2\pi$, there is a finite number of singular directions.

To each singular direction $d$ is attached a Stokes matrix $St_d = (c_{ij})$ with $c_{ii} = 1$ and $c_{ij} \neq 0$ only if $q = q_i - q_j \in Q$ and $d \in Fr(q)$.

It is a unipotent matrix.

At least for many confluent hypergeometric equations this theorem leads to an effective computation of the $g$ and $G$. 

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The Lie-algebra \( \mathfrak{g} \) of \( G \).

**General assumption** : \( \mathfrak{g} \) is semi-simple (no general criterion in the local case).

**Crucial fact** : the adjoint action of \( T_e \) on \( \mathfrak{g} \) is known.

The characters can be identified to \( \mathbb{Q} \) : 
\( q = q_i - q_j \) gives

\[
\tau = \text{diag}(\lambda_1, \cdots, \lambda_n) \mapsto \frac{\lambda_i}{\lambda_j}
\]
The corresponding decomposition in weight spaces is

$$\mathfrak{g} = \mathfrak{g}_0(T_e) \bigoplus_{q \in \mathcal{Q}} \mathfrak{g}_q.$$ 

Each Stokes matrix $St_d$ is unipotent. Its logarithm is an element $\dot{\Delta}_d = (s_{ij})$. For $d \in Fr(q)$, let $\dot{\Delta}_{d,q} = (\delta_{ij})$ with $\delta_{ij} = s_{ij}$ if $q_i - q_j = q$ and $\delta_{ij} = 0$ otherwise. If $d \notin Fr(q)$, let $\dot{\Delta}_{d,q} = 0$. Then

$$\dot{\Delta}_d = \bigoplus_{q \in \mathcal{Q}} \dot{\Delta}_{d,q}.$$ 

One can prove that $\dot{\Delta}_{d,q}$, named *alien derivative* in direction $d$ of weight $q$, belongs to $\mathfrak{g}_q$. 
Let $M_s$ and $M_u$ be the semi-simple and unipotent part of $M$. The logarithm of $M_u$ is denoted by $m_u$. The closed group $D$ generated by $M_s$ is diagonal so $D = T_m \times F$ where $F$ is finite and $T_m$ is a torus named by Ramis monodromy torus.

Then (Ramis)

$$g_0(T_e) = \mathcal{L}(T_m) + \mathcal{L}(T_e) + \mathbb{C}m_u + \sum_{q \in \mathbb{Q}} [g_q, g_{-q}].$$
The Weyl group is isomorphic to \( \frac{N_G(T_{\text{max}})}{T_{\text{max}}} \)
where \( T_{\text{max}} \) is a maximal torus.

Very often \( T_e \) is not a maximal torus:
– \( T_f = T_m \times T_e \) (the formal torus) can be larger than \( T_e \),
– \( \dim g_q > 1 \) is possible.

Let \( a \) be a non trivial weight for the adjoint representation of a torus \( T \) on \( g \) (semi-simple) s.t. \( \dim g_a(T) > 1 \). There exists a semi-simple element \( H_a \in [g_a(T), g_{a-1}(T)] \) such that, in restriction to \( g_a(T) \), it has at least two distinct eigenvalues.
From weight space to root space

Start with a torus $T$ acting on $\mathfrak{g}$.

**Step 0.** In the decomposition

$$\mathfrak{g} = \mathfrak{g}_0(T) \bigoplus_{a \in \chi(T) \setminus \{1\}} \mathfrak{g}_a(T). \quad (2)$$

1. if for all $a \in \chi(T) \setminus \{1\}$, dim $\mathfrak{g}_a(T) = 1$, STOP

2. ELSE apply several times if necessary
Step 1. For $a$ s.t. $\dim g_a(T) > 1$, choose $H_a \in [g_a(T), g_{a-1}(T)]$ as before. Let

$$\mathcal{L}_1 = \mathcal{L}(T) \oplus \mathbb{C}H_a$$

acts on $\mathfrak{g}$. It stabilizes each weight space of $T$. In particular $g_a(T)$ is a direct sum of $m > 1$ weight spaces for $\mathcal{L}_1$:

$$g_a(T) = g_{b_1}(\mathcal{L}_1) \oplus \cdots \oplus g_{b_m}(\mathcal{L}_1)$$

where

$$b_1|_{\mathcal{L}(T)} = \cdots = b_m|_{\mathcal{L}(T)} = \bar{a}$$

(weight of $\mathcal{L}(T)$ corresponding to $a$) and

$$1 \leq \dim g_{b_i}(\mathcal{L}_1) < \dim g_a(T)$$
Thus the adjoint representation of $\mathcal{L}_1$ on $g$ writes:

$$g = \mathcal{g}_0(\mathcal{L}_1) \bigoplus_{b \in \mathcal{L}_1^*} \mathcal{g}_b(\mathcal{L}_1) \quad (3)$$

with $\mathcal{g}_0(\mathcal{L}_1) \subset \mathcal{g}_0(T)$ and each weight space of (2) is a direct sum of weight spaces appearing in (3).

At least for one space ($\mathcal{g}_a(T)$) there are two components.

If for all $b$, $\dim \mathcal{g}_b(\mathcal{L}_1) = 1$, STOP, ELSE GO TO STEP1.
At each step the dimension of at least one of the spaces decreases so the process terminates with a decomposition of $g$ of the form

$$g = g_0(\mathcal{L}_p) \bigoplus_{b \in L_p^*} g_b(L_p)$$

with $g_0(\mathcal{L}_p) \subset g_0(\mathcal{L}_{p-1}) \subset \cdots \subset g_0(T)$ and each weight space $g_b(\mathcal{L}_p)$ has dimension 1 and thus is a root space $g_\alpha$ of $g$, with $\alpha \in R(\mathfrak{h})$.

We apply this process with $T = T_e$ and when $\dim g_q > 1$, we try an element $H_q = [\hat{\Delta}_{d,q}, \hat{\Delta}_{d',-q}]$ with $d \in Fr(q)$ and $d' \in Fr(-q)$. 

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If $g_0(\mathfrak{L}_p)$ is a Cartan sub-algebra ($= \mathfrak{h}$), then (3) is the usual adjoint representation of $g$ and $W$ can be described from it. In particular, if $g_0(T_e)$ is a Cartan sub-algebra, then also $g_0(\mathfrak{L}_p) = \mathfrak{h}$. This leads to the final question.

**Regularity of the exponential torus**

Recall that $T$ is regular if its centralizer $C_G(T)$ is a maximal torus.

As $M \in C_G(T_e)$, we see that a necessary condition for $T_e$ to be regular is $M_u = 1$ (or $M$ semi-simple).
Known fact:
If $G$ is reductive and $S$ is a singular torus

$$C_g(S) = \mathfrak{L}(T) \bigoplus g_\beta \text{ and } S = \bigcap_{\beta \in L} (\ker \beta)^0$$

where $T \supset S$ is a maximal torus and, if $R$ is the associated root system,

$$L = \{ \beta \in R \mid S \subset (\ker \beta)^0 \}.$$ 

using this, one can prove

1. If the $n$ determinant factors are all distincts then $T_e$ is regular.
2. $T_e$ is regular iff there is a basis of

$$V = \bigoplus_{i=1}^{s} V_{q_i}$$

$(s = \#D)$ in which each matrix in $g$, $X = (X_{ij})_{1 \leq i, j \leq s}$ blocked after this decomposition, satisfies: the $s$ matrices $X_{ii}$ are diagonal.