

Local differential Galois group and adjoint representation.

E. Compoint and A. Duval
U.S.T.Lille

Hoboken April 14, 2007

The differential Galois group of a linear differential operator or, equivalently, a linear differential system is an *algebraic* group G reflecting properties of the operator.

One would like to give an interpretation of the Weyl group W of G , known to give important informations in the algebraic theory of groups, in terms of the differential operator.

Here we deal with the local situation where one can take advantage of Ramis' description of the differential Galois group.

The local group G .

Let

$$Y' = A(x)Y \quad (1)$$

be a linear differential system of dimension n , given by a $n \times n$ matrix A with entries in the field $\mathbb{C}(\{x\})$ of meromorphic convergent power series in the complex variable x . The differential Galois group is

$$G = \text{Aut}_{diff}(K, \mathbb{C}(\{x\}))$$

where K is any Picard-Vessiot extension of $\mathbb{C}(\{x\})$ associated to the system.

As Picard-Vessiot extension one can use the field generated by a *formal* fundamental matrix of solutions of the form

$$\hat{Y} = F(x)x^L e^{Q(t)}$$

where $F \in GL_n(\mathbb{C}((x)))$ is an invertible matrix of meromorphic formal series, $L \in gl_n(\mathbb{C})$ is a constant matrix and $Q(t)$ is a diagonal matrix of polynomials (without constant terms) in the variable t where $t^\nu = x$ for some $\nu \in \mathbb{N}^*$.

One can describe G by its action on the dimension n vector space V of formal solutions of the system (1). The space V is spanned by the columns of \hat{Y} and G appears as a closed subgroup of $GL_n(\mathbb{C})$.

Its Lie-algebra \mathfrak{g} is then a sub-Lie-algebra of $gl_n(\mathbb{C})$ on which G acts (adjoint action).

Let $\mathcal{D} = \{q \mid q \text{ is one entry of } Q\}$ be the set of *determinant factors*. Then

$$V = \bigoplus_{q \in \mathcal{D}} V_q$$

where V_q is the subspace of solutions the exponential part of which is e^q .

Theorem 1 (Ramis). *The group G is topologically generated by the exponential torus, the formal monodromy and the Stokes matrices.*

The *formal monodromy* is the element γ of G acting on V by the matrix M such that

$$\hat{Y}(e^{2i\pi}x) = \hat{Y}(x)M.$$

It permutes the V_q : $\gamma(V_q) = V_{\gamma(q)}$.

The *exponential torus* T_e reflects the integral relations between the determinant factors.

A $\tau \in T_e$ acts via a matrix of the form

$$\bigoplus_{q \in \mathcal{D}} \lambda_q I_{r_q}$$

where $r_q = \dim V_q$ and $\lambda_q \in \mathbb{C}^*$. Among them $\sigma = \dim \bigoplus_{q \in \mathcal{D}} q\mathbb{Z}$ are free, the others are

monomials reflecting multiplicatively the additive relations among the determinant factors.

Thus T_e is a *torus* of dimension σ .

Let $\mathcal{Q} = \{q_i - q_j \mid q_i, q_j \in \mathcal{D}, q_i \neq q_j\}$.

A direction d is *singular* if for some $q \in \mathcal{Q}$, e^q has maximal decay in direction d .

One then writes $d \in Fr(q)$.

Modulo 2π , there is a finite number of singular directions.

To each singular direction d is attached a *Stokes matrix* $St_d = (c_{ij})$ with $c_{ii} = 1$ and $c_{ij} \neq 0$ only if $q = q_i - q_j \in \mathcal{Q}$ and $d \in Fr(q)$. It is a *unipotent* matrix.

At least for many confluent hypergeometric equations this theorem leads to an effective computation of the \mathfrak{g} and G .

The Lie-algebra \mathfrak{g} of G .

General assumption : \mathfrak{g} is semi-simple (no general criterion in the local case).

Crucial fact : the adjoint action of T_e on \mathfrak{g} is known.

The characters can be identified to \mathcal{Q} :
 $q = q_i - q_j$ gives

$$\tau = \text{diag}(\lambda_1, \dots, \lambda_n) \longmapsto \frac{\lambda_i}{\lambda_j}$$

The corresponding decomposition in weight spaces is

$$\mathfrak{g} = \mathfrak{g}_0(T_e) \bigoplus_{q \in \mathcal{Q}} \mathfrak{g}_q.$$

Each Stokes matrix St_d is unipotent.

Its logarithm is an element $\dot{\Delta}_d = (s_{ij})$.

For $d \in Fr(q)$, let $\dot{\Delta}_{d,q} = (\delta_{ij})$ with $\delta_{ij} = s_{ij}$ if $q_i - q_j = q$ and $\delta_{ij} = 0$ otherwise.

If $d \notin Fr(q)$, let $\dot{\Delta}_{d,q} = 0$. Then

$$\dot{\Delta}_d = \bigoplus_{q \in \mathcal{Q}} \dot{\Delta}_{d,q}.$$

One can prove that $\dot{\Delta}_{d,q}$, named *alien derivative* in direction d of weight q , belongs to \mathfrak{g}_q .

Let M_s and M_u be the semi-simple and unipotent part of M . The logarithm of M_u is denoted by m_u . The closed group D generated by M_s is diagonal so $D = T_m \times F$ where F is finite and T_m is a torus named by Ramis *monodromy torus*.

Then (Ramis)

$$\mathfrak{g}_0(T_e) = \mathfrak{L}(T_m) + \mathfrak{L}(T_e) + \mathbb{C}m_u + \sum_{q \in \mathcal{Q}} [\mathfrak{g}_q, \mathfrak{g}_{-q}].$$

The Weyl group is isomorphic to $\frac{N_G(T_{max})}{T_{max}}$ where T_{max} is a maximal torus.

Very often T_e is not a maximal torus :

- $T_f = T_m \times T_e$ (the *formal torus*) can be larger than T_e ,
- $\dim \mathfrak{g}_q > 1$ is possible.

Let a be a non trivial weight for the adjoint representation of a torus T on \mathfrak{g} (semi-simple) s.t. $\dim \mathfrak{g}_a(T) > 1$. There exists a semi-simple element $H_a \in [\mathfrak{g}_a(T), \mathfrak{g}_{a-1}(T)]$ such that, in restriction to $\mathfrak{g}_a(T)$, it has at least two distinct eigenvalues.

From weight space to root space

Start with a torus T acting on \mathfrak{g} .

Step 0. In the decomposition

$$\mathfrak{g} = \mathfrak{g}_0(T) \oplus \bigoplus_{a \in \chi(T) \setminus \{1\}} \mathfrak{g}_a(T). \quad (2)$$

1. if for all $a \in \chi(T) \setminus \{1\}$, $\dim \mathfrak{g}_a(T) = 1$,
STOP
2. ELSE apply several times if necessary

Step 1. For a s.t. $\dim \mathfrak{g}_a(T) > 1$, choose $H_a \in [\mathfrak{g}_a(T), \mathfrak{g}_{a-1}(T)]$ as before. Let

$$\mathcal{L}_1 = \mathcal{L}(T) \oplus \mathbb{C}H_a$$

acts on \mathfrak{g} . It stabilizes each weight space of T . In particular $\mathfrak{g}_a(T)$ is a direct sum of $m > 1$ weight spaces for \mathcal{L}_1 :

$$\mathfrak{g}_a(T) = \mathfrak{g}_{b_1}(\mathcal{L}_1) \oplus \cdots \oplus \mathfrak{g}_{b_m}(\mathcal{L}_1)$$

where

$$b_1|_{\mathcal{L}(T)} = \cdots = b_m|_{\mathcal{L}(T)} = \bar{a}$$

(weight of $\mathcal{L}(T)$ corresponding to a) and

$$1 \leq \dim \mathfrak{g}_{b_i}(\mathcal{L}_1) < \dim \mathfrak{g}_a(T)$$

Thus the adjoint representation of \mathfrak{L}_1 on \mathfrak{g} writes :

$$\mathfrak{g} = \mathfrak{g}_0(\mathfrak{L}_1) \bigoplus_{b \in \mathfrak{L}_1^*} \mathfrak{g}_b(\mathfrak{L}_1) \quad (3)$$

with $\mathfrak{g}_0(\mathfrak{L}_1) \subset \mathfrak{g}_0(T)$ and each weight space of (2) is a direct sum of weight spaces appearing in (3).

At least for one space ($\mathfrak{g}_a(T)$) there are two components.

If for all b , $\dim \mathfrak{g}_b(\mathfrak{L}_1) = 1$, STOP, ELSE GO TO STEP1.

At each step the dimension of at least one of the spaces decreases so the process terminates with a decomposition of \mathfrak{g} of the form

$$\mathfrak{g} = \mathfrak{g}_0(\mathcal{L}_p) \bigoplus_{b \in L_p^*} \mathfrak{g}_b(L_p)$$

with $\mathfrak{g}_0(\mathcal{L}_p) \subset \mathfrak{g}_0(\mathcal{L}_{p-1}) \subset \cdots \subset \mathfrak{g}_0(T)$ and each weight space $\mathfrak{g}_b(\mathcal{L}_p)$ has dimension 1 and thus is a root space \mathfrak{g}_α of \mathfrak{g} , with $\alpha \in R(\mathfrak{h})$.

We apply this process with $T = T_e$ and when $\dim \mathfrak{g}_q > 1$, we try an element $H_q = [\dot{\Delta}_{d,q}, \dot{\Delta}_{d',-q}]$ with $d \in Fr(q)$ and $d' \in Fr(-q)$.

If $\mathfrak{g}_0(\mathcal{L}_p)$ is a Cartan sub-algebra ($= \mathfrak{h}$), then (3) is the usual adjoint representation of \mathfrak{g} and W can be described from it.

In particular, if $\mathfrak{g}_0(T_e)$ is a Cartan sub-algebra, then also $\mathfrak{g}_0(\mathcal{L}_p) = \mathfrak{h}$. This leads to the final question.

Regularity of the exponential torus

Recall that T is regular if its centralizer $C_G(T)$ is a maximal torus.

As $M \in C_G(T_e)$, we see that a necessary condition for T_e to be regular is $M_u = 1$ (or M semi-simple).

Known fact :

If G is reductive and S is a singular torus

$$C_{\mathfrak{g}}(S) = \mathfrak{L}(T) \bigoplus_{\beta \in L} \mathfrak{g}_{\beta} \text{ and } S = \bigcap_{\beta \in L} (\ker \beta)^{\circ}$$

where $T \supset S$ is a maximal torus and, if R is the associated root system,

$$L = \{\beta \in R \mid S \subset (\ker \beta)^{\circ}\}.$$

using this, one can prove

1. If the n determinant factors are all distincts then T_e is regular.

2. T_e is regular iff there is a basis of

$$V = \bigoplus_{i=1}^s V_{q_i}$$

($s = \#\mathcal{D}$) in which each matrix in \mathfrak{g} , $X = (X_{ij})_{1 \leq i, j \leq s}$ blocked after this decomposition, satisfies : the s matrices X_{ii} are diagonal.

Ref. : E. C.; A. D. *Groupe de Galois différentiel local et représentation adjointe* Ann. Fac. Sc. Toulouse (to appear).