Borel-Laplace summation of q-series and confluence

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Let $q \in \mathbb{R}$, $\boxed{0 < q < 1}$ **and** $\boxed{[n]_q^! = \prod_{i=1}^{n} \frac{1-q^i}{1-q}}$**:**

$$\mathcal{B}_q : x\mathbb{C}[[x]] \to \mathbb{C}[[\xi]], \qquad \sum_{n\geq 0} a_n x^{n+1} \mapsto \sum_{n\geq 0} \frac{a_n}{[n]_q^!} \xi^n.$$

The "q-Euler series" $\boxed{\mathcal{E}_q(x) = \sum_{n\geq 0} (-1)^n [n]_q^! x^{n+1}}$ is convergent!

Which is the relation between $\mathcal{E}_q(x)$ and the sum of the Euler series $\hat{E}(x) = \sum_{n\geq 0} (-1)^n n! x^{n+1}$?

Theorem.

Suppose that:

- $y(q,x) \in x\mathbb{C}[[x]]$ *for* $q \in (\eta, 1]$, $\eta \in (0,1)$;
- $\phi(q,\xi) = \mathcal{B}_q y(q,x) \in \mathbb{C}\{\xi\}$ *is solution of a fuchsian non resonant operator at* ∞, *satisfying assumptions* $(\star)$ *(cf. next slide);*
- *there exists a direction* $d \in [0, 2\pi)$ *such that* $\phi(q,\xi)$ *is holomorphic in a domain containing the* $e^{id}\mathbb{R}^+$.

$$\Rightarrow \quad \lim_{q \to 1^-} y(q,x) = \mathcal{L}^d\left(\phi(1,\xi)\right) = \int_0^{\infty e^{id}} \phi(1,\xi) e^{-\frac{-\xi}{x}} d\xi \,.$$

uniformly on any compact of $V := \{|\arg x - d| < \frac{\pi}{2}\}$.

Rmk. $\phi(q,\xi)$, for $q \neq 1$, is the germ at 0 of a meromorphic function on $\mathbb{C}$ (with $q \in \mathbb{R}$!).

- $\phi(1,\xi)$ is solution of $\sum_{i=0}^{\mu} A_i(1,\xi)\delta^i\phi(1,\xi)=0$, $\delta=\xi\frac{d}{d\xi}$, $A_i(1,\xi)\in\mathbb{C}[\xi]$, fuchsian non resonant at ∞.
- $\phi(q,\xi)$, $q\in(\eta,1)$, is solution of $\sum_{i=0}^{\mu} A_i(q,\xi)\delta_q^i\phi(q,\xi)=0$, with $\delta_q(f)=\frac{f(q\xi)-f(\xi)}{q-1}$, $A_i(q,\xi)\in\mathbb{C}[\xi]$, fuchsian at ∞.
- The Newton polygon at ∞ is independent of $q\in(\eta,1]$.
- the coefficients $A_i(q,\xi)$ tends uniformly to $A_i(1,x)$ when $q\longrightarrow 1$, on any compact of $\mathbb{P}^1_{\mathbb{C}}$ where they are defined.
- For any q sufficiently closed to 1 there exists a constant gauge transformation $C(q)\in Gl_\mu(\mathbb{C})$ such that the constant term at ∞ of the matrix

$$C(q)^{-1}\left(\begin{array}{c|ccc} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ \hline -\frac{A_0(q,x)}{A_\mu(q,x)} & -\frac{A_1(q,x)}{A_\mu(q,x)} & \dots & -\frac{A_{\mu-1}(q,x)}{A_\mu(q,x)} \end{array}\right)C(q)$$

is in the Jordan normal form. We suppose that for $q\in(\eta,1]$ the entries of the matrix $C(q)$ are continuous functions of q and that the form of the Jordan blocks is independent of q.

Corollary.

Let $y(x) = \sum_{n\geq 0} y_n x^{n+1} \in x\mathbb{C}[[x]]$ be a Gevrey series of order one such that $\phi(\xi) = \mathcal{B}_1 y(x)$ is solution of a differential equation $\sum_{i=0}^{\mu} A_i(x)\delta^i y = 0$, fuchsian and non resonant at ∞.
Then the series $\mathbf{y}_q(x)$, solution of $\sum_{i=0}^{\mu} A_i(x)\delta_q^i y = 0$, that converges coefficientwise to $y(x)$, converges uniformly to the Borel sum of $y(x)$ on the compacts of a convenient sector $V = \{|\arg x - d| < \pi/2\}$.

Corollary.

Let $\mathcal{E}_q(x)$ be the analytic continuation of $\sum_{n\geq 1}(-1)[n]_q^! x^{n+1}$, $\mathcal{E}(x)$ the sum of $\hat{E} = \sum_{n\geq 1}(-1)n! x^{n+1}$.

For any $x \in \mathbb{C} \smallsetminus (-\infty, 0]$

$$\lim_{q\to 1^-} \mathcal{E}_q(x) = \mathcal{E}(x)\,.$$

The convergence is uniform over any compact of $\mathbb{C} \smallsetminus (-\infty, 0]$.

More precisely, $\forall \epsilon \in (0, \pi)$ and $R > 0$ $\exists K_{\epsilon,R} > 0$ such that $\forall x \in V_{\epsilon;R} := \{x \in \mathbb{C}^* : |\arg x| \leq \pi - \epsilon, |x| > R\}$ and $\forall q \in (0, 1)$:

$$|\mathcal{E}_q(x) - \mathcal{E}(x)| \leq 2(1-q)\Big[\,|\log x| + K_{\epsilon,R}\,\Big]\,.$$

$a, b \in \mathbb{C}$, $a - b \notin \mathbb{Z}$; $(a)_n = a(a+1)\cdots(a+n-1)$;
$(q^a; q) = (1-q^a)(1-q^{a+1})\cdots(1-q^{a+n-1})$.

Corollary.

The analytic function

$$ {}_2\Phi_1\left(q^a, q^b; -; q, \frac{x}{1-q}\right) := \sum_{n\geq 0} \frac{(q^a;q)_n (q^b;q)_n}{(q^n;q)_n} \left(\frac{x}{1-q}\right)^n $$

converges uniformly to the Borel sum of the confluent hypergeometric series

$$ {}_2F_1(a, b; -; x) = \sum_{n\geq 0} \frac{(a)_n (b)_n}{n!} x^n $$

on the compacts of a convenient sector centered at 0, *when* $q \to 1^-$.

Let $q \in \mathbb{R}$, $q > 1$:

There are two q-analogues of $n!$:

$$q^{n(n-1)/2} \longrightarrow \lim_{q\to 1} q^{n(n-1)/2} = 1.$$

$$[n]_q^! = \prod_{i=1}^{n} \frac{1-q^i}{1-q} \longrightarrow \lim_{q\to 1} [n]_q^! = n!$$

but $\lim_{n\to\infty} q^{-n(n-1)/2}[n]_q^! = 1$.

This gives rise to many (at least 4) different kind of summations: a discrete and a continuous one for each kind of momenta.

What can we say on the relations among them?

$p = q^{-1}$

$\theta(x) = \sum_{n\in\mathbb{Z}} q^{-n(n-1)/2} x^n$

$e_q(x) = \sum_{n\geq 0} \frac{x}{[n]_q^!}$

$\int_{[\lambda]} f(x)\, \frac{d_p x}{x} = (1-p) \sum_{n\in\mathbb{Z}} f(\lambda p^n)$

$* * *$

$\hat{f} = \sum_{n\geq 0} f_n x^{n+1}$, s.t.

$$\phi(\xi) = \sum_{n\geq 0} \frac{f_n}{q^{n(n-1)/2}} \xi^n \in \mathbb{C}\{\xi\}$$

$$\psi(\xi) = \sum_{n\geq 0} \frac{f_n}{[n]_q^!} \xi^n \in \mathbb{C}\{\xi\}.$$

$$\begin{array}{ccc}
\phi & \longrightarrow & f^d := \frac{q}{\ln q} \int_0^{e^{id}\infty} \frac{\phi(\xi)}{\theta(q\frac{\xi}{x})} d\xi \\
\downarrow & & \\
f^{[\lambda]} := \frac{q}{1-p} \int_{[\lambda]} \frac{\phi(\xi)}{\theta(q\frac{\xi}{x})} d_p\xi & & \mathrm{f}^d := \frac{q-1}{\ln q} \int_0^{e^{id}\infty} \frac{\psi(\xi)}{e_q(q\frac{\xi}{x})} d\xi \\
 & & \uparrow \\
\mathrm{f}^{[\lambda]} := q \int_{[\lambda]} \frac{\psi(\xi)}{e_q(q\frac{\xi}{x})} d_p\xi & \longleftarrow & \psi
\end{array}$$

Theorem.

If $\hat{\mathrm{f}}$ is solution of a linear q-difference equation having only one slope at 0 equal to 1, then $f^{[\lambda]} = \mathrm{f}^{[\lambda]}$ and $f^d = \mathrm{f}^d$.