Symplectic Properties of the Space of Fuchsian Equations in the Moduli Space of Logarithmic Connections

Jonathan Aidan

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AMS Special Session on Differential Algebra
April 15, 2007
Introduction: Two Classical Worlds

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<tr>
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Linear Differential Operators over $\mathbb{P}^1_C$

- of order $n$
- with singular locus $\subset S$

Linear Differential Systems over $\mathbb{P}^1_C$

- of rank $n$
- with singular locus $\subset S$

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**Symplectic Properties of the Space of Fuchsian Equations in the Moduli Space of Logarithmic Connections**

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- Locally from $\mathcal{E}_0$ into $\mathcal{N}_0$
- From $\mathcal{N}_0$ into $\mathcal{M}_0$

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## Linear Differential Operators

- **Order**: $n$
- **Domain**: $\mathbb{P}^1_C$
- **Isomorphism**: $E_0 \cong \mathcal{M}_0$
- **Type**: Fuchsian

## Linear Differential Systems

- **Rank**: $n$
- **Domain**: $\mathbb{P}^1_C$
- **Isomorphism**: $\nabla_0 \cong \mathcal{N}_0$
- **Type**: Logarithmic

### Question

Can one find some explanation of this doubling? We give a "symplectic explanation" of it.
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**Linear Differential Operators**
- of order $n$
- with singular locus $\subset S$
- Fuchsian
- with non-resonant exponents

**Linear Differential Systems**
- of rank $n$
- with singular locus $\subset S$
- Logarithmic
- with non-resonant residues

**At an irreducible point,**
- dimension $= g$
- where $g$ is some integer depending on $n$ and $|S|$. 

**Questions**
- Can one find some explanation of this doubling?
- We give a "symplectic explanation" of it.
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We give a “symplectic explanation” of it.
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The Starting Point $L_0$

**Basic Setting**

*Fix $n \geq 1$. Fix $m \geq 1$, and fix*

$$S := \{s_0, \ldots, s_m, s_{m+1} = \infty\} \subset \mathbb{P}^1_C \text{ such that } s_j \neq s_j.$$

Then we set $P = \prod_{j=0}^{m} (z - s_j)$ and $P_0 = \prod_{j=1}^{m} (z - s_j)$. 

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The Starting Point \( L_0 \)

**Basic Setting**

*Fix \( n \geq 1\). Fix \( m \geq 1\), and fix*

\[
S := \{s_0, \ldots, s_m, s_{m+1} = \infty\} \subset \mathbb{P}^1_C \text{ such that } s_j \neq s_j'
\]

Then we set \( P = \prod_{j=0}^{m}(z - s_j) \) and \( P_0 = \prod_{j=1}^{m}(z - s_j) \).

**Definition of \( \mathcal{E} \)**

\[
\mathcal{E} = \{L \in W = \mathbb{C}(z)[d/dz], \text{ } L \text{ is a monic Fuchsian operator of order } n \text{ with singular locus in } S\}.
\]

These \( L \)'s are exactly those that can be written as:

\[
L = \left( \frac{d}{dz} \right)^n + \frac{a_{n-1}}{P} \left( \frac{d}{dz} \right)^{n-1} + \cdots + \frac{a_0}{P^n} \text{ with } a_i \in \mathbb{C}[Z]_{\leq i \cdot m}
\]
The Starting Point $L_0$

**Basic Setting**

*Fix $n \geq 1$. Fix $m \geq 1$, and fix*

$$S := \{s_0, \ldots, s_m, s_{m+1} = \infty\} \subset \mathbb{P}^1_C \text{ such that } s_j \neq s_{j'}$$

*Then we set* $P = \prod_{j=0}^{m}(z - s_j)$ *and* $P_0 = \prod_{j=1}^{m}(z - s_j)$.

**Definition of $\mathcal{E}$**

$$\mathcal{E} = \{L \in W = \mathbb{C}(z)[d/dz], L \text{ is a monic Fuchsian operator of order } n \text{ with singular locus in } S\}.$$  

*These $L$'s are exactly those that can be written as:*

$$L = (\frac{d}{dz})^n + \frac{a_{n-1}}{P}(\frac{d}{dz})^{n-1} + \cdots + \frac{a_0}{P^n} \text{ with } a_i \in \mathbb{C}[Z]_{\leq i \cdot m}$$

**Definition of $L_0$**

*We fix $L_0 \in \mathcal{E}$. Suppose $L_0$ is irreducible and each of its local monodromies has distinct eigenvalues.*
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Summary Diagram

$\mathcal{E} \ni L_0$
Deformation Space of $L_0$ with Fixed Local Monodromies

Consider the Riemann scheme

$$\mathcal{P}_0 := \left( \begin{array}{cccc} s_0 & \cdots & s_m & \infty \\ \{e_{0i}\}_{i=1\ldots n} & \cdots & \{e_{mi}\}_{i=1\ldots n} & \{e_{\infty i}\}_{i=1\ldots n} \end{array} \right)$$

of $L_0$. For each $j$, $\{e_{ji}\}_{i=1\ldots n}$ are the roots of the indicial polynomial of $L_0$ at $s_j$. 

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Deformation Space of $L_0$ with Fixed Local Monodromies

Consider the Riemann scheme

$$
P_0 := \begin{pmatrix}
  s_0 & \cdots & s_m & \infty \\
  \{ e_{0i} \}_{i=1\ldots n} & \cdots & \{ e_{mi} \}_{i=1\ldots n} & \{ e_{\infty i} \}_{i=1\ldots n}
\end{pmatrix}
$$

of $L_0$. For each $j$, $\{ e_{ji} \}_{i=1\ldots n}$ are the roots of the indicial polynomial of $L_0$ at $s_j$.

Definition of $\mathcal{E}_0$

$$
\mathcal{E}_0 = \{ L \in \mathcal{E}, \text{ the Riemann scheme of } L \text{ is } P_0 \}
$$

This is an affine variety of dimension

$$
dim(\mathcal{E}_0) = g := m \frac{n(n-1)}{2} - (n - 1)
$$
Deformation Space of $L_0$ with Fixed Local Monodromies

Consider the Riemann scheme

$$P_0 := \begin{pmatrix} s_0 & \cdots & s_m \\ \{ e_{0i} \}_{i=1\ldots n} & \cdots & \{ e_{mi} \}_{i=1\ldots n} & \{ e_{\infty i} \}_{i=1\ldots n} \end{pmatrix}$$

of $L_0$. For each $j$, $\{ e_{ji} \}_{i=1\ldots n}$ are the roots of the indicial polynomial of $L_0$ at $s_j$.

**Definition of $E_0$**

$$E_0 = \{ L \in E, \text{ the Riemann scheme of } L \text{ is } P_0 \}$$

This is an affine variety of dimension

$$\dim(E_0) = g := m \frac{n(n-1)}{2} - (n-1)$$

By hypothesis, each $\{ e_{ji} \}_{i=1\ldots n}$ is a family of non-congruent complex numbers modulo $\mathbb{Z}$. Therefore, we lie “non-resonant” case and any $L \in E_0$ has the same local monodromies as $L_0$. $E_0$ is the connected component of $L_0$ in the space of monic Fuchsian operators of order $n$ with singular locus in $S$ and same local monodromies as $L_0$. 
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Summary Diagram

$\mathcal{E} \ni L_0 \longrightarrow \mathcal{E}_0$
From an Operator to a Connection

To any \( L \in \mathcal{E} \), \( L = \left( \frac{d}{dz} \right)^n + \frac{a_{n-1}}{P} \left( \frac{d}{dz} \right)^{n-1} + \cdots + \frac{a_0}{P^n} \) with \( a_i \in \mathbb{C}[z]_{\leq i \cdot m} \), one can attach the connection:

\[
\nabla : (\mathcal{O}_{\mathbb{P}^1})^n \rightarrow (\mathcal{O}_{\mathbb{P}^1})^n \otimes \Omega^1
\]

defined by:

\[
\nabla|_{\mathbb{P}^1 \setminus \{\infty\}} = d - A dz
\]

where

\[
A = \begin{pmatrix}
\frac{-a_0}{P^n} \\
1 & \cdots \\
\cdots & \ddots \\
1 & \cdots & \frac{-a_{n-1}}{P}
\end{pmatrix} \in M_n(\mathbb{C}(z))
\]

**Question**

Can one find some gauge transformation that transforms \( \nabla \) into a logarithmic connection, i.e. of the form

\[
\sum_{j=0}^{m} \frac{A_j}{z-s_j} \, dz \quad \text{with constant matrices } A_j?
\]
### Definition of $\mathcal{N}$

\[
\mathcal{N} = \left\{ (A_0, A_1, \ldots, A_m, A_\infty) \in M_n(\mathbb{C})^{m+2}, \sum_{j=0}^{\infty} A_j = 0, \right\}
\]

\[
A_0 = \begin{pmatrix}
* & 0 & 0 & 0 \\
1 & * & 0 & 0 \\
0 & \cdots & \cdots & 0 \\
0 & 0 & 1 & *
\end{pmatrix}, \quad A_j(1 \leq j \leq m) = \begin{pmatrix}
* & * & * & * \\
0 & * & * & * \\
0 & 0 & \cdots & \cdots \\
0 & 0 & 0 & *
\end{pmatrix}
\]

### Theorem 1 (van der Put & Singer [3])

For any $L \in \mathcal{E}$, there exists $(A_0, \ldots, A_m, A_\infty) \in \mathcal{N}$ s.t. $L$ is the minimal monic operator in $W$ which annihilates the global section $e_1 = (1, 0, \ldots, 0) \in (O_{\mathbb{P}^1})^n$ of the logarithmic connection $d - \sum_{j=0}^{m} A_j \frac{dz}{z-s_j}$. 

\[\text{Definition of } \nabla_0 \]

We fix $\nabla_0 = (A_0, \ldots, A_m, A_\infty) \in \mathcal{N}$, one of the possible tuples for $L_0$ given by Theorem 1.
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**Definition of $\mathcal{N}$**

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A_0 = \begin{pmatrix}
* & 0 & 0 & 0 \\
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0 & 0 & 1 & *
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A_j (1 \leq j \leq m) = \begin{pmatrix}
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\end{pmatrix} \}$$

**Theorem 1 (van der Put & Singer [3])**

*For any $L \in \mathcal{E}$, there exists $(A_0, \ldots, A_m, A_{\infty}) \in \mathcal{N}$ s.t. $L$ is the minimal monic operator in $W$ which annihilates the global section $e_1 = (1, 0 \ldots, 0) \in (\mathcal{O}_{\mathbb{P}^1})^n$ of the logarithmic connection $d - \sum_{j=0}^{m} A_j \frac{dz}{z-s_j}$.***

**Definition of $\nabla_0$**

*We fix $\nabla_0 = (A^0_0, \ldots, A^0_m, A^0_{\infty}) \in \mathcal{N}$, one of the possible tuples for $L_0$ given by Theorem 1.*
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Summary Diagram

$\mathcal{N} \ni \nabla_0$

$\varphi (Thm.1)$

$E \ni L_0 \rightarrow E_0$
Constructing $O$

For any $j \in \{0, \ldots, \infty\}$, denote by $O_j$ the conjugacy class of $A_j^0$:

$$O_j = \text{GL}_n(\mathbb{C}).A_j^0$$

and set

$$O = O_0 \times \cdots \times O_m \times O_{\infty}$$
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Space of Deformations of $\nabla_0$ as Log. Connection

Construction of $\mathcal{O}$

For any $j \in \{0, \ldots, \infty\}$, denote by $\mathcal{O}_j$ the conjugacy class of $A_j^0$:

$$\mathcal{O}_j = \text{GL}_n(\mathbb{C}).A_j^0$$

and set

$$\mathcal{O} = \mathcal{O}_0 \times \cdots \times \mathcal{O}_m \times \mathcal{O}_\infty$$

Definition of the symplectic form $\omega$ on $\mathcal{O}$

$\mathcal{O}$ is (almost) canonically endowed with the symplectic form $\omega \in \Omega^2_{\mathcal{O}}$:

$$\forall A = (A_0, \ldots, A_\infty) \in \mathcal{O}, \quad \forall (B^1, B^2) \in (T_A \mathcal{O})^2,$$

$$\omega_A(B^1, B^2) = \sum_{j=0}^{\infty} \text{Tr}(A_j[U_j^1, U_j^2]),$$

where the $U_j^k$'s satisfy

$$\forall k = 1, 2 \quad \forall j = 0 \ldots \infty, \quad B_j^k = [U_j^k, A_j]$$
**Definition of $\mathcal{M}_0$**

We apply \textbf{symplectic reduction} to some open neighborhood $\mathcal{V}_0 \subset \mathcal{O}$ of $\nabla_0$

$$\mathcal{V}_0 \subset \{(A_0, \ldots, A_\infty), \cap_{j=0}^m \text{Com}(A_j) = \mathbb{C} \times I_n\}$$

for diagonal adjoint action of $\text{GL}_n(\mathbb{C})$ and moment map

$$\Phi : (A_0, \ldots, A_\infty) \mapsto \sum_{j=0}^{\infty} A_j$$

at its regular value 0:

$$\mathcal{V}_0 \cap \Phi^{-1}(\{0\}) \xrightarrow{\pi} \mathcal{M}_0 := \Phi^{-1}(\{0\})/\text{GL}_n(\mathbb{C}).$$
**Definition of $\mathcal{M}_0$**

*We apply symplectic reduction to some open neighborhood $\mathcal{V}_0 \subset \mathcal{O}$ of $\nabla_0$*

\[ \mathcal{V}_0 \subset \{(A_0, \ldots, A_\infty), \cap_{j=0}^m \text{Com}(A_j) = \mathbb{C} \times I_n\} \]

*for diagonal adjoint action of $GL_n(\mathbb{C})$ and moment map*

\[ \Phi : (A_0, \ldots, A_\infty) \mapsto \sum_{j=0}^{\infty} A_j \]

*at its regular value 0:*

\[ \mathcal{V}_0 \cap \Phi^{-1}(\{0\}) \xrightarrow{\pi} \mathcal{M}_0 := \Phi^{-1}(\{0\}) / GL_n(\mathbb{C}). \]

Then, $\mathcal{M}_0$ is the local **moduli space of deformations of $\nabla_0$ as logarithmic connection with the same local monodromies as $\nabla_0$ (and thereby, as $L_0$).**

The form $\omega$ induces on $\mathcal{M}_0$ a structure of symplectic complex manifold, and $\text{dim}(\mathcal{M}_0) = 2g$. 
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Summary Diagram
For any $j \in \{0, \ldots, m\}$, let $(\lambda_{1j}, \ldots, \lambda_{nj}) \in \mathbb{C}^n$ be the diagonal of the matrix $A_j^0$.

**Definition of $\mathcal{N}_0$**

\[
\mathcal{N}_0 := \left\{ (A_0, \ldots, A_\infty) \in \mathcal{N} \cap \mathcal{O} \mid (j=1, \ldots, m) \right\}
\]

\[
A_0 = \begin{pmatrix}
\lambda_{1,0} & 0 & 0 & 0 \\
1 & \lambda_{2,0} & 0 & 0 \\
0 & \ddots & \ddots & 0 \\
0 & 0 & 1 & \lambda_{n,0}
\end{pmatrix},
A_j = \begin{pmatrix}
\lambda_{1j} & * & * & * \\
0 & \lambda_{2j} & * & * \\
0 & 0 & \ddots & \vdots \\
0 & 0 & 0 & \lambda_{nj}
\end{pmatrix}
\]
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Embedding of \((\mathcal{E}_0, L_0)\) in \((\mathcal{N}_0, \nabla_0)\)

Following van der Put & Singer, we introduce the map

\[
\varphi : \mathcal{N}_0 \to W, \varphi(\nabla) := \left(\frac{1}{P}\right)^n L_n(\nabla),
\]

where \(L_n(\nabla)\) is defined as follows. For any \(\nabla = (A_0, \ldots, A_m, A_\infty) \in \mathcal{N}_0\), let \((A_{x,y})_{x \leq y}\) be the polynomials in \(\mathbb{C}[z]\) defined by

\[
\sum_{j=0}^{m} \frac{A_j}{z - s_j} = \frac{1}{P} \begin{pmatrix}
A_{1,1} & zA_{1,2} & \cdots & zA_{1,n} \\
P_0 & A_{2,2} & \ddots & \vdots \\
0 & \ddots & \ddots & zA_{n-1,n} \\
0 & 0 & \ldots & P_0 & A_{n,n}
\end{pmatrix}
\]
Let \( M_i = Pd/dz - A_{i,i} - (i - 1)zP_0' \), \( L_0(\nabla) = 1 \) and define \( L_k(\nabla) \in \mathbb{C}[z][P \cdot d/dz], k \geq 1 \), by

\[ L_k(\nabla) = M_k L_{k-1}(\nabla) - P A_{k-1,i} L_{k-2}(\nabla) - PP_0 A_{k,k-1} L_{k-3}(\nabla) - \cdots - PP_0^{i-2} A_{k,1} L_0(\nabla) \]

Then (Thm. 1), \( \varphi(\nabla) \) is the minimal monic annihilator in \( W \) of \( e_1 \) for the logarithmic connection \( d - \sum_{j=0}^{m} A_j \frac{dz}{z-s_j} \), and

**Proposition 1**

- \( \varphi(\text{Connected component of } \nabla_0 \text{ in } \mathcal{N}_0) \subset \mathcal{E}_0 \)
- \( T_{\nabla_0} \varphi : T_{\nabla_0} \mathcal{N}_0 \rightarrow T_{L_0} \mathcal{E}_0 \) is an isomorphism. We can therefore consider

\[ \psi = \varphi^{-1} : (\mathcal{E}_0, L_0) \rightarrow (\mathcal{N}_0, \nabla_0) \]
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**Proposition 2**

- \(\dim(\mathcal{N}_0) = \frac{1}{2} \dim(\mathcal{M}_0)\) (which is equal to \(g\)).
- For any \(\nabla \in \mathcal{N}_0\), \(\text{GL}_n(\mathbb{C}).\nabla \cap \mathcal{N}_0 = \{\nabla\}\).
- \(\mathcal{N}_0(\subset \Phi^{-1}(\{0\}) \cap \mathcal{V}_0)\) embeds in \(\mathcal{M}_0\) via \(\pi\).
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Summary Diagram
Via these local embeddings, we may view the tangent space of $\mathcal{E}_0$ at $L_0$ as a subspace of the tangent space of $\mathcal{M}_0$ at $\nabla_0$, and we have:

**Theorem 2**

$(\mathcal{E}_0, L_0)$ is a Lagrangian submanifold of $(\mathcal{M}_0, \nabla_0)$ (via $\pi \circ \psi$).

N.B.: related results by Dubrovin-Mazzocco [2] (Darboux-coordinates on $\mathcal{M}$), and by S. Szabo [4] (Katz’s question on Hodge structures).

The dimension of $\mathcal{E}_0$ is half the dimension of $\mathcal{M}_0$ so it is sufficient to prove that $\mathcal{N}_0$ is an isotropic submanifold of $\mathcal{O}$ relatively to $\omega$. The proof of the latter statement consists in checking that the local contribution of each singularity to the expression $\omega_A(B^1, B^2) = \sum_{j=0}^{\infty} \text{Tr}(A_j[U^1_j, U^2_j])$ vanishes. The only difficulty lies at $\infty$, which is dealt with as follows.
Proposition

Let $\nabla = (*, \ldots, *, A_\infty) \in \mathcal{N}_0$, let $(*, \ldots, *, B_\infty) \in T_{\nabla} \mathcal{N}_0$ and let $U \in M_n(\mathbb{C})$ satisfy $B_\infty = [U, A_\infty]$. Then, there exists a strictly upper-triangular matrix $U_\infty \in T^+$ such that $B_\infty = [U_\infty, A_\infty]$.

This follows from the fact that for $k \in \mathbb{N}$, $A^k_\infty$ is lower-triangular of order $k$, with $(-1)^k$ on the $k$-th diagonal, together with

Lemma

Same hypotheses as in Proposition 3. Let $e = (e_1, \ldots, e_n)$ be the standard basis of $E := \mathbb{C}^n$, and let $a, b, u \in \text{End}(E)$, represented in $e$ by $A_\infty, B_\infty, U$. Let $\{E_i, i = 1, \ldots, n\}$ be the flag associated to $e$, and set $E_{-1} = E_0 = \{0\}$. Finally, consider the map $\chi : \{1, \ldots, n\} \rightarrow \{0, \ldots, n\}$ which attaches to $i$ the unique integer $\chi(i) \in \{0, \ldots, n\}$ s.t. $u(e_i) \in E_{\chi(i)} \setminus E_{\chi(i)-1}$. Then,

- $\forall i \in \{1, \ldots, n - 1\}$, $a(E_i \setminus E_{i-1}) \subset E_{i+1} \setminus E_i$.
- If $\chi \leq n - 1$, then $\forall i = 1, \ldots, n - 1$, $\chi(i) \leq n - 2$.

