Symplectic Properties of the Space of Fuchsian Equations in the Moduli Space of Logarithmic Connections

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Introduction: Two Classical Worlds

Linear Differential Operators over $\mathbb{P}^1_{\mathbb{C}}$	Linear Differential Systems over $\mathbb{P}^1_{\mathbb{C}}$

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Question

Can one find some explanation of this doubling?

We give a "symplectic explanation" of it.

Goal

$$\begin{tikzcd}
 & \mathcal{M}_0 \\
\mathcal{E}_0 \arrow[ur, hook] &
\end{tikzcd}$$

The Starting Point L_0

Basic Setting

Fix $n \geq 1$*. Fix* $m \geq 1$*, and fix*

$$S := \{s_0, \dots, s_m, s_{m+1} = \infty\} \subset \mathbb{P}^1_C \text{ such that } s_j \neq s_{j'}$$

Then we set $P = \prod_{j=0}^{m}(z - s_j)$ and $P_0 = \prod_{j=1}^{m}(z - s_j)$.

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Definition of $\mathcal{E}$

$\mathcal{E} = \{L \in W = \mathbb{C}(z)[d/dz]$, *$L$ is a monic Fuchsian operator of order n with singular locus in S*$\}$.

These L's are exactly those that can be written as:
$L = (\frac{d}{dz})^n + \frac{a_{n-1}}{P}(\frac{d}{dz})^{n-1} + \cdots + \frac{a_0}{P^n}$ with $a_i \in \mathbb{C}[z]_{\leq i \cdot m}$

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Definition of L_0

We fix $L_0 \in \mathcal{E}$. Suppose L_0 is ***irreducible*** *and* ***each of its local monodromies has distinct eigenvalues****.*

Summary Diagram

$\mathcal{E} \ni L_0$

Deformation Space of L_0 with Fixed Local Monodromies

Consider the Riemann scheme

$$\mathcal{P}_0 := \left(\begin{array}{cccc} s_0 & \cdots & s_m & \infty \\ \{e_{0i}\}_{i=1\ldots n} & \cdots & \{e_{mi}\}_{i=1\ldots n} & \{e_{\infty i}\}_{i=1\ldots n} \end{array} \right)$$

of L_0. For each j, $\{e_{ji}\}_{i=1\ldots n}$ are the roots of the indicial polynomial of L_0 at s_j.

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Definition of $\mathcal{E}_0$

$\mathcal{E}_0 = \{L \in \mathcal{E}$, *the Riemann scheme of L is* $\mathcal{P}_0\}$

This is an affine variety of dimension

$$dim(\mathcal{E}_0) = g := m\frac{n(n-1)}{2} - (n-1)$$

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By hypothesis, each $\{e_{ji}\}_{i=1\ldots n}$ is a family of non-congruent complex numbers modulo $\mathbb{Z}$. Therefore, we lye "non-resonant" case and any $L \in \mathcal{E}_0$ has the same local monodromies as L_0. **$\mathcal{E}_0$ is the connected component of L_0 in the space of monic Fuchsian operators of order n with singular locus in S and same local monodromies as L_0.**

Summary Diagram

$$\begin{tikzcd}
\mathcal{E} \ni L_0 \arrow[r, dashed] & \mathcal{E}_0
\end{tikzcd}$$

From an Operator to a Connection

To any $L \in \mathcal{E}$, $L = (\frac{d}{dz})^n + \frac{a_{n-1}}{P}(\frac{d}{dz})^{n-1} + \cdots + \frac{a_0}{P^n}$ with $a_i \in \mathbb{C}[z]_{\leq i \cdot m}$, one can attach the connection:

$$\nabla : (\mathcal{O}_{\mathbb{P}^1})^n \to (\mathcal{O}_{\mathbb{P}^1})^n \otimes \Omega^1$$

defined by:

$$\nabla|_{\mathbb{P}^1 \setminus \{\infty\}} = d - Adz$$

where

$$A = \begin{pmatrix} & & & -\frac{a_0}{P^n} \\ 1 & & & \vdots \\ & \ddots & & \\ & & 1 & -\frac{a_{n-1}}{P} \end{pmatrix} \in M_n(\mathbb{C}(z))$$

Question

Can one find some gauge transformation that transforms ∇ into a logarithmic connection, i.e. of the form $\sum_{j=0}^{m} \frac{A_j}{z-s_j} dz$ with constant matrices A_j?

Definition of $\mathcal{N}$

$$\mathcal{N} = \{(A_0, A_1, \ldots, A_m, A_\infty) \in M_n(\mathbb{C})^{m+2}, \sum_{j=0}^{\infty} A_j = 0,$$

$$A_0 = \begin{pmatrix} * & 0 & 0 & 0 \\ 1 & * & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & * \end{pmatrix}, A_{j(1\leq j\leq m)} = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & * \end{pmatrix} \}$$

Theorem 1 (van der Put & Singer [3])

For any $L \in \mathcal{E}$, there exists $(A_0, \ldots, A_m, A_\infty) \in \mathcal{N}$ s.t. L is the minimal monic operator in W which annihilates the global section $e_1 = (1, 0 \ldots, 0) \in (\mathcal{O}_{\mathbb{P}^1})^n$ of the logarithmic connection $d - \sum_{j=0}^{m} A_j \frac{dz}{z-s_j}$.

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Definition of ∇_0

We fix $\nabla_0 = (A_0^0, \ldots, A_m^0, A_\infty^0) \in \mathcal{N}$, one of the possible tuples for L_0 given by Theorem 1.

Summary Diagram

$$\begin{tikzcd}
\mathcal{N} \ni \nabla_0 \arrow[d, bend left, "\varphi(\mathit{Thm}.1)"] & \\
\mathcal{E} \ni L_0 \arrow[u, dashed, bend left] \arrow[r, dashed] & \mathcal{E}_0
\end{tikzcd}$$

Space of Deformations of ∇_0 as Log. Connection

Construction of $\mathcal{O}$

For any $j \in \{0, \ldots, \infty\}$, denote by $\mathcal{O}_j$ the conjugacy class of A_j^0: $\mathcal{O}_j = GL_n(\mathbb{C}).A_j^0$ and set

$$\mathcal{O} = \mathcal{O}_0 \times \cdots \times \mathcal{O}_m \times \mathcal{O}_\infty$$

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Definition of the symplectic form ω on $\mathcal{O}$

$\mathcal{O}$ is (almost) canonically endowed with the symplectic form $\omega \in \Omega^2_{\mathcal{O}}$:

$$\forall A = (A_0, \dots, A_\infty) \in \mathcal{O}, \quad \forall (B^1, B^2) \in (T_A\mathcal{O})^2,$$

$$\omega_A(B^1, B^2) = \sum_{j=0}^{\infty} Tr(A_j[U_j^1, U_j^2]),$$

where the U_j^k's satisfy

$$\forall k = 1, 2 \quad \forall j = 0 \dots \infty, \quad B_j^k = [U_j^k, A_j]$$

Definition of $\mathcal{M}_0$

We apply ***symplectic reduction*** *to some open neighborhood* $\mathcal{V}_0 \subset \mathcal{O}$ *of* ∇_0

$$\mathcal{V}_0 \subset \{(A_0, \dots, A_\infty), \cap_{j=0}^{m} Com(A_j) = \mathbb{C}^\times I_n\}$$

for diagonal adjoint action of $GL_n(\mathbb{C})$ *and moment map*

$$\Phi : (A_0, \dots, A_\infty) \mapsto \sum_{j=0}^{\infty} A_j$$

at its regular value 0*:*

$$\mathcal{V}_0 \cap \Phi^{-1}(\{0\}) \overset{\pi}{\twoheadrightarrow} \mathcal{M}_0 := \Phi^{-1}(\{0\})/GL_n(\mathbb{C}).$$

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Then, $\mathcal{M}_0$ is the local **moduli space of deformations of** ∇_0 **as logarithmic connection with the same local monodromies as** ∇_0 (and thereby, as L_0).

The form ω induces on $\mathcal{M}_0$ a structure of symplectic complex manifold, and $dim(\mathcal{M}_0) = 2g$.

Summary Diagram

$$\begin{tikzcd}
 & & \mathcal{O} \\
\mathcal{N} \ni \nabla_0 \arrow[urr, dashed] \arrow[ddd, bend left, two heads, "\varphi(Thm.1)"] & & \mathcal{V}_0 \cap \Phi^{-1}(\{0\}) \arrow[u, "\subseteq"] \arrow[d, two heads, "\pi"] \\
 & & \mathcal{M}_0 \\
\mathcal{E} \ni L_0 \arrow[uuu, dashed, bend left] \arrow[r, dashed] & \mathcal{E}_0 &
\end{tikzcd}$$

For any $j \in \{0, \ldots, m\}$, let $(\lambda_{1j}, \ldots, \lambda_{nj}) \in \mathbb{C}^n$ be the diagonal of the matrix A_j^0.

Definition of $\mathcal{N}_0$

$$\mathcal{N}_0 := \Big\{ (A_0, \ldots, A_\infty) \in \mathcal{N} \cap \mathcal{O} /_{(j=1,\ldots,m)}$$

$$A_0 = \begin{pmatrix} \lambda_{1,0} & 0 & 0 & 0 \\ 1 & \lambda_{2,0} & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & \lambda_{n,0} \end{pmatrix}, A_j = \begin{pmatrix} \lambda_{1j} & * & * & * \\ 0 & \lambda_{2j} & * & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_{nj} \end{pmatrix} \Big\}$$

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Embedding of $(\mathcal{E}_0, L_0)$ in $(\mathcal{N}_0, \nabla_0)$

Following van der Put & Singer, we introduce the map

$$\varphi : \mathcal{N}_0 \to W \ , \varphi(\nabla) := (\frac{1}{P})^n L_n(\nabla),$$

where $L_n(\nabla)$ is defined as follows. For any $\nabla = (A_0, \dots, A_m, A_\infty) \in \mathcal{N}_0$, let $(A_{x,y})_{x \leq y}$ be the polynomials in $\mathbb{C}[z]$ defined by

$$\sum_{j=0}^{m} \frac{A_j}{z - s_j} = \frac{1}{P} \begin{pmatrix} A_{1,1} & zA_{1,2} & \cdots & zA_{1,n} \\ P_0 & A_{2,2} & \ddots & \vdots \\ 0 & \ddots & \ddots & zA_{n-1,n} \\ 0 & 0 & \dots P_0 & A_{n,n} \end{pmatrix}$$

Let $M_i = Pd/dz - A_{i,i} - (i-1)zP_0'$, $L_0(\nabla) = 1$ and define $L_k(\nabla) \in \mathbb{C}[z][P \cdot d/dz], k \geq 1$, by

$$L_k(\nabla) = M_k L_{k-1}(\nabla) - PA_{k-1,i} L_{k-2}(\nabla) -$$

$$- PP_0 A_{k,k-1} L_{k-3}(\nabla) - \cdots - PP_0^{i-2} A_{k,1} L_0(\nabla)$$

Then (Thm. 1), $\varphi(\nabla)$ is the minimal monic annihilator in W of e_1 for the logarithmic connection $d - \sum_{j=0}^{m} A_j \frac{dz}{z-s_j}$, and

Proposition 1

- *φ(Connected component of ∇_0 in $\mathcal{N}_0$) $\subset \mathcal{E}_0$*
- *$T_{\nabla_0}\varphi : T_{\nabla_0}\mathcal{N}_0 \to T_{L_0}\mathcal{E}_0$ is an isomorphism. We can therefore consider*

$$\psi = \varphi^{-1} : (\mathcal{E}_0, L_0) \to (\mathcal{N}_0, \nabla_0)$$

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\mathcal{E} \ni L_0 \arrow[r, dashed] \arrow[u, dashed, bend left] & \mathcal{E}_0 \arrow[u, bend left, "\psi=\varphi^{-1}(Prop.1)"] & \mathcal{M}_0
\end{tikzcd}$$

Embedding of $(\mathcal{N}_0, \nabla_0)$ into $(\mathcal{M}_0, \nabla_0)$

Proposition 2

- *dim*$(\mathcal{N}_0) = \frac{1}{2}$*dim*$(\mathcal{M}_0)$ *(which is equal to g).*
- *For any* $\nabla \in \mathcal{N}_0$, $GL_n(\mathbb{C}).\nabla \cap \mathcal{N}_0 = \{\nabla\}$.
- $\mathcal{N}_0(\subset \Phi^{-1}(\{0\}) \cap \mathcal{V}_0)$ *embeds in* $\mathcal{M}_0$ *via* π.

Summary Diagram

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\mathcal{E} \ni L_0 \arrow[r, dashed] \arrow[uuu, dashed] & \mathcal{E}_0 \arrow[uu, hook, bend left, "\psi=\varphi^{-1}(Prop.1)"] \arrow[ur, hook, "\pi\circ\psi"] &
\end{tikzcd}$$

Via these local embeddings, we may view the tangent space of $\mathcal{E}_0$ at L_0 as a subspace of the tangent space of $\mathcal{M}_0$ at ∇_0, and we have:

Theorem 2

$(\mathcal{E}_0, L_0)$ *is a Lagrangian submanifold of* $(\mathcal{M}_0, \nabla_0)$ *(via* $\pi \circ \psi$*).*

N.B.: related results by Dubrovin-Mazzocco [2] (Darboux-coordinates on $\mathcal{M}$), and by S. Szabo [4] (Katz's question on Hodge structures).

The dimension of $\mathcal{E}_0$ is half the dimension of $\mathcal{M}_0$ so it is sufficient to prove that $\mathcal{N}_0$ *is an isotropic submanifold of* $\mathcal{O}$ *relatively to* ω. The proof of the latter statement consists in checking that the local contribution of each singularity to the expression $\omega_A(B^1, B^2) = \sum_{j=0}^{\infty} Tr(A_j[U_j^1, U_j^2])$ vanishes. The only difficulty lies at ∞, which is dealt with as follows.

Proposition

Let $\nabla = (*, \dots, *, A_\infty) \in \mathcal{N}_0$, *let* $(*, \dots, *, B_\infty) \in T_\nabla \mathcal{N}_0$ *and let* $U \in M_n(\mathbb{C})$ *satisfy* $B_\infty = [U, A_\infty]$. *Then, there exists a strictly upper-triangular matrix* $U_\infty \in \mathcal{T}^+$ *such that* $B_\infty = [U_\infty, A_\infty]$.

This follows from the fact that for $k \in \mathbb{N}$, A_∞^k is lower-triangular of order k, with $(-1)^k$ on the k-th diagonal, together with

Lemma

Same hypotheses as in Proposition 3. Let $e = (e_1, \dots, e_n)$ *be the standard basis of* $E := \mathbb{C}^n$, *and let* $a, b, u \in End(E)$, *represented in e by* A_∞, B_∞, U. *Let* $\{E_i, i = 1, ..., n\}$ *be the flag associated to e, and set* $E_{-1} = E_0 = \{0\}$. *Finally, consider the map* $\chi : \{1, \dots, n\} \to \{0, \dots, n\}$ *which attaches to i the unique integer* $\chi(i) \in \{0, \dots, n\}$ *s.t.* $u(e_i) \in E_{\chi(i)} \setminus E_{\chi(i)-1}$. *Then,*

- $\forall i \in \{1, \dots, n-1\}$, $a(E_i \setminus E_{i-1}) \subset E_{i+1} \setminus E_i$.
- *If* $\chi \leq n-1$, *then* $\forall i = 1, .., n-1, \chi(i) \leq n-2$

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