

Symplectic Properties of the Space of Fuchsian Equations in the Moduli Space of Logarithmic Connections

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of order n with singular locus $\subset S$	of rank n with singular locus $\subset S$

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At an irreducible point, dimension = g	At an irreducible point, dimension = $2g$

where g is some integer depending on n and $|S|$.

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Question

Can one find some explanation of this doubling?

We give a “symplectic explanation” of it.

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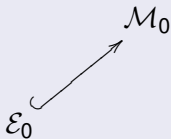
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Goal



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The Starting Point L_0

Basic Setting

Fix $n \geq 1$. Fix $m \geq 1$, and fix

$$S := \{s_0, \dots, s_m, s_{m+1} = \infty\} \subset \mathbb{P}_C^1 \text{ such that } s_j \neq s_{j'}$$

Then we set $P = \prod_{j=0}^m (z - s_j)$ and $P_0 = \prod_{j=1}^m (z - s_j)$.

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Definition of \mathcal{E}

$\mathcal{E} = \{L \in W = \mathbb{C}(z)[d/dz], L \text{ is a monic Fuchsian operator of order } n \text{ with singular locus in } S\}$.

These L 's are exactly those that can be written as:

$$L = \left(\frac{d}{dz}\right)^n + \frac{a_{n-1}}{P} \left(\frac{d}{dz}\right)^{n-1} + \dots + \frac{a_0}{P^n} \text{ with } a_i \in \mathbb{C}[z]_{\leq i \cdot m}$$

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Definition of L_0

We fix $L_0 \in \mathcal{E}$. Suppose L_0 is **irreducible** and **each of its local monodromies has distinct eigenvalues**.

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$$\mathcal{E} \ni L_0$$

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Deformation Space of L_0 with Fixed Local Monodromies

Consider the Riemann scheme

$$\mathcal{P}_0 := \left(\begin{array}{ccc} s_0 & \cdots & s_m & \infty \\ \{e_{0i}\}_{i=1\dots n} & \cdots & \{e_{mi}\}_{i=1\dots n} & \{e_{\infty i}\}_{i=1\dots n} \end{array} \right)$$

of L_0 . For each j , $\{e_{ji}\}_{i=1\dots n}$ are the roots of the indicial polynomial of L_0 at s_j .

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Definition of \mathcal{E}_0

$$\mathcal{E}_0 = \{L \in \mathcal{E}, \text{ the Riemann scheme of } L \text{ is } \mathcal{P}_0\}$$

This is an affine variety of dimension

$$\dim(\mathcal{E}_0) = g := m \frac{n(n-1)}{2} - (n-1)$$

Deformation Space of L_0 with Fixed Local Monodromies

Consider the Riemann scheme

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By hypothesis, each $\{e_{ji}\}_{i=1\dots n}$ is a family of non-congruent complex numbers modulo \mathbb{Z} . Therefore, we have “non-resonant” case and any $L \in \mathcal{E}_0$ has the same local monodromies as L_0 .

\mathcal{E}_0 is the connected component of L_0 in the space of monic Fuchsian operators of order n with singular locus in S and same local monodromies as L_0 .

Summary Diagram

$$\mathcal{E} \ni L_0 \text{ --- } \triangleright \mathcal{E}_0$$

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From an Operator to a Connection

To any $L \in \mathcal{E}$, $L = \left(\frac{d}{dz}\right)^n + \frac{a_{n-1}}{P} \left(\frac{d}{dz}\right)^{n-1} + \dots + \frac{a_0}{P^n}$ with $a_j \in \mathbb{C}[z]_{\leq j \cdot m}$, one can attach the connection:

$$\nabla : (\mathcal{O}_{\mathbb{P}^1})^n \rightarrow (\mathcal{O}_{\mathbb{P}^1})^n \otimes \Omega^1$$

defined by:

$$\nabla|_{\mathbb{P}^1 \setminus \{\infty\}} = d - Adz$$

where

$$A = \begin{pmatrix} & & -\frac{a_0}{P^n} & \\ & & \vdots & \\ 1 & & & \\ & \ddots & & \\ & & 1 & -\frac{a_{n-1}}{P} \end{pmatrix} \in M_n(\mathbb{C}(z))$$

Question

Can one find some gauge transformation that transforms ∇ into a logarithmic connection, i.e. of the form

$$\sum_{j=0}^m \frac{A_j}{z-s_j} dz \text{ with constant matrices } A_j?$$

Definition of \mathcal{N}

$$\mathcal{N} = \left\{ (A_0, A_1, \dots, A_m, A_\infty) \in M_n(\mathbb{C})^{m+2}, \sum_{j=0}^{\infty} A_j = 0, \right. \\ \left. A_0 = \begin{pmatrix} * & 0 & 0 & 0 \\ 1 & * & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & * \end{pmatrix}, A_{j(1 \leq j \leq m)} = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & * \end{pmatrix} \right\}$$

Theorem 1 (van der Put & Singer [3])

For any $L \in \mathcal{E}$, there exists $(A_0, \dots, A_m, A_\infty) \in \mathcal{N}$ s.t. L is the minimal monic operator in W which annihilates the global section $e_1 = (1, 0, \dots, 0) \in (\mathcal{O}_{\mathbb{P}^1})^n$ of the logarithmic connection $d - \sum_{j=0}^m A_j \frac{dz}{z-s_j}$.

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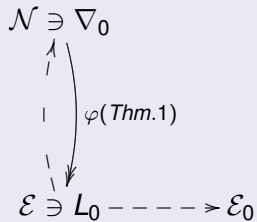
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Definition of ∇_0

We fix $\nabla_0 = (A_0^0, \dots, A_m^0, A_\infty^0) \in \mathcal{N}$, one of the possible tuples for L_0 given by Theorem 1.

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Construction of \mathcal{O}

For any $j \in \{0, \dots, \infty\}$, denote by \mathcal{O}_j the conjugacy class of A_j^0 :
 $\mathcal{O}_j = GL_n(\mathbb{C}) \cdot A_j^0$ and set

$$\mathcal{O} = \mathcal{O}_0 \times \dots \times \mathcal{O}_m \times \mathcal{O}_\infty$$

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Definition of the symplectic form ω on \mathcal{O}

\mathcal{O} is (almost) canonically endowed with the symplectic form
 $\omega \in \Omega_{\mathcal{O}}^2$:

$$\forall A = (A_0, \dots, A_\infty) \in \mathcal{O}, \quad \forall (B^1, B^2) \in (T_A \mathcal{O})^2,$$

$$\omega_A(B^1, B^2) = \sum_{j=0}^{\infty} \text{Tr}(A_j [U_j^1, U_j^2]),$$

where the U_j^k 's satisfy

$$\forall k = 1, 2 \quad \forall j = 0 \dots \infty, \quad B_j^k = [U_j^k, A_j]$$

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Definition of \mathcal{M}_0

We apply **symplectic reduction** to some open neighborhood $\mathcal{V}_0 \subset \mathcal{O}$ of ∇_0

$$\mathcal{V}_0 \subset \{(A_0, \dots, A_\infty), \cap_{j=0}^m \text{Com}(A_j) = \mathbb{C}^\times I_n\}$$

for diagonal adjoint action of $GL_n(\mathbb{C})$ and moment map

$$\Phi : (A_0, \dots, A_\infty) \mapsto \sum_{j=0}^{\infty} A_j$$

at its regular value 0:

$$\mathcal{V}_0 \cap \Phi^{-1}(\{0\}) \xrightarrow{\pi} \mathcal{M}_0 := \Phi^{-1}(\{0\})/GL_n(\mathbb{C}).$$

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Then, \mathcal{M}_0 is the local **moduli space of deformations of ∇_0 as logarithmic connection with the same local monodromies as ∇_0** (and thereby, as L_0).

The form ω induces on \mathcal{M}_0 a structure of symplectic complex manifold, and $\dim(\mathcal{M}_0) = 2g$.

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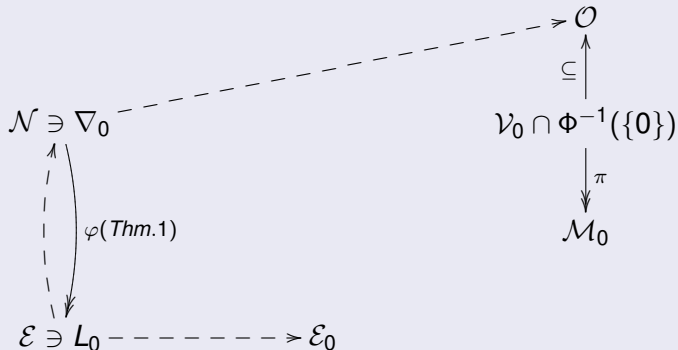
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Summary Diagram



For any $j \in \{0, \dots, m\}$, let $(\lambda_{1j}, \dots, \lambda_{nj}) \in \mathbb{C}^n$ be the diagonal of the matrix A_j^0 .

Definition of \mathcal{N}_0

$$\mathcal{N}_0 := \{(A_0, \dots, A_\infty) \in \mathcal{N} \cap \mathcal{O} / (j=1, \dots, m)$$

$$A_0 = \begin{pmatrix} \lambda_{1,0} & 0 & 0 & 0 \\ 1 & \lambda_{2,0} & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & \lambda_{n,0} \end{pmatrix}, A_j = \begin{pmatrix} \lambda_{1j} & * & * & * \\ 0 & \lambda_{2j} & * & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_{nj} \end{pmatrix} \}$$

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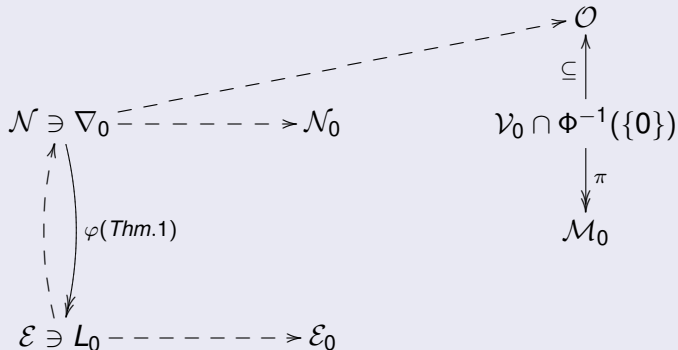
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Summary Diagram



Embedding of (\mathcal{E}_0, L_0) in $(\mathcal{N}_0, \nabla_0)$

Following van der Put & Singer, we introduce the map

$$\varphi : \mathcal{N}_0 \rightarrow W, \varphi(\nabla) := \left(\frac{1}{P}\right)^n L_n(\nabla),$$

where $L_n(\nabla)$ is defined as follows. For any $\nabla = (A_0, \dots, A_m, A_\infty) \in \mathcal{N}_0$, let $(A_{x,y})_{x \leq y}$ be the polynomials in $\mathbb{C}[z]$ defined by

$$\sum_{j=0}^m \frac{A_j}{z - s_j} = \frac{1}{P} \begin{pmatrix} A_{1,1} & zA_{1,2} & \cdots & zA_{1,n} \\ P_0 & A_{2,2} & \ddots & \vdots \\ 0 & \ddots & \ddots & zA_{n-1,n} \\ 0 & 0 & \dots & P_0 & A_{n,n} \end{pmatrix}$$

Let $M_i = Pd/dz - A_{i,i} - (i-1)zP'_0$, $L_0(\nabla) = 1$ and define $L_k(\nabla) \in \mathbb{C}[z][P \cdot d/dz]$, $k \geq 1$, by

$$L_k(\nabla) = M_k L_{k-1}(\nabla) - P A_{k-1,i} L_{k-2}(\nabla) - \\ - P P_0 A_{k,k-1} L_{k-3}(\nabla) - \dots - P P_0^{i-2} A_{k,1} L_0(\nabla)$$

Then (Thm. 1), $\varphi(\nabla)$ is the minimal monic annihilator in W of e_1 for the logarithmic connection $d - \sum_{j=0}^m A_j \frac{dz}{z-s_j}$, and

Proposition 1

- $\varphi(\text{Connected component of } \nabla_0 \text{ in } \mathcal{N}_0) \subset \mathcal{E}_0$
- $T_{\nabla_0} \varphi : T_{\nabla_0} \mathcal{N}_0 \rightarrow T_{L_0} \mathcal{E}_0$ is an isomorphism. We can therefore consider

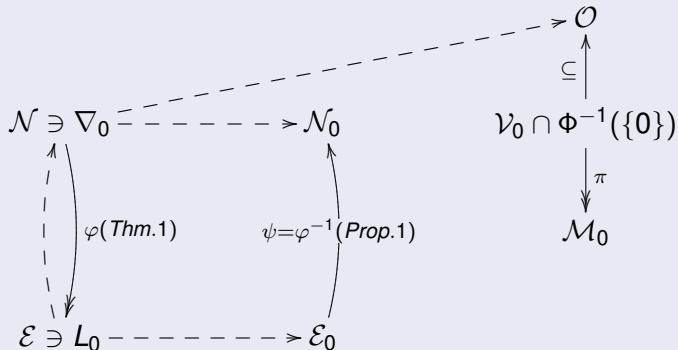
$$\psi = \varphi^{-1} : (\mathcal{E}_0, L_0) \rightarrow (\mathcal{N}_0, \nabla_0)$$

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Proposition 2

- $\dim(\mathcal{N}_0) = \frac{1}{2} \dim(\mathcal{M}_0)$ (which is equal to g).
- For any $\nabla \in \mathcal{N}_0$, $GL_n(\mathbb{C}) \cdot \nabla \cap \mathcal{N}_0 = \{\nabla\}$.
- $\mathcal{N}_0 \subset \Phi^{-1}(\{0\}) \cap \mathcal{V}_0$ embeds in \mathcal{M}_0 via π .

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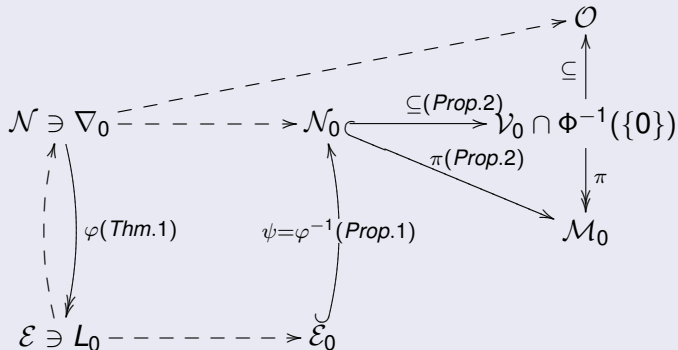
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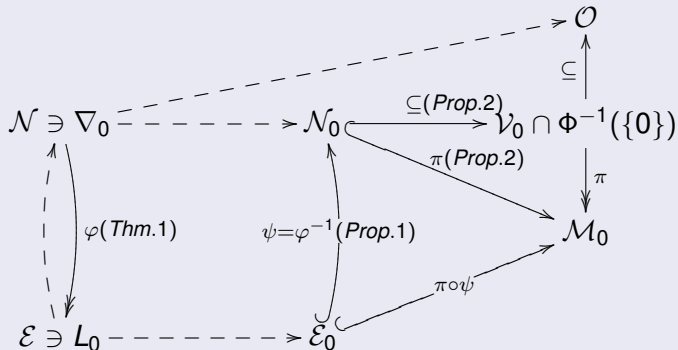
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Via these local embeddings, we may view the tangent space of \mathcal{E}_0 at L_0 as a subspace of the tangent space of \mathcal{M}_0 at ∇_0 , and we have:

Theorem 2

(\mathcal{E}_0, L_0) is a Lagrangian submanifold of $(\mathcal{M}_0, \nabla_0)$ (via $\pi \circ \psi$).

N.B.: related results by Dubrovin-Mazzocco [2] (Darboux-coordinates on \mathcal{M}), and by S. Szabo [4] (Katz's question on Hodge structures).

The dimension of \mathcal{E}_0 is half the dimension of \mathcal{M}_0 so it is sufficient to prove that \mathcal{N}_0 is an isotropic submanifold of \mathcal{O} relatively to ω . The proof of the latter statement consists in checking that the local contribution of each singularity to the expression $\omega_A(B^1, B^2) = \sum_{j=0}^{\infty} \text{Tr}(A_j[U_j^1, U_j^2])$ vanishes. The only difficulty lies at ∞ , which is dealt with as follows.

Proposition


Let $\nabla = (*, \dots, *, A_\infty) \in \mathcal{N}_0$, let $(*, \dots, *, B_\infty) \in T_\nabla \mathcal{N}_0$ and let $U \in M_n(\mathbb{C})$ satisfy $B_\infty = [U, A_\infty]$. Then, there exists a strictly upper-triangular matrix $U_\infty \in \mathcal{T}^+$ such that $B_\infty = [U_\infty, A_\infty]$.


This follows from the fact that for $k \in \mathbb{N}$, A_∞^k is lower-triangular of order k , with $(-1)^k$ on the k -th diagonal, together with


Lemma

Same hypotheses as in Proposition 3. Let $e = (e_1, \dots, e_n)$ be the standard basis of $E := \mathbb{C}^n$, and let $a, b, u \in \text{End}(E)$, represented in e by A_∞, B_∞, U . Let $\{E_i, i = 1, \dots, n\}$ be the flag associated to e , and set $E_{-1} = E_0 = \{0\}$. Finally, consider the map $\chi : \{1, \dots, n\} \rightarrow \{0, \dots, n\}$ which attaches to i the unique integer $\chi(i) \in \{0, \dots, n\}$ s.t. $u(e_i) \in E_{\chi(i)} \setminus E_{\chi(i)-1}$. Then,

- $\forall i \in \{1, \dots, n-1\}$, $a(E_i \setminus E_{i-1}) \subset E_{i+1} \setminus E_i$.
- If $\chi \leq n-1$, then $\forall i = 1, \dots, n-1$, $\chi(i) \leq n-2$

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