Non-integrability of some Hamiltonians with rational potentials

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Introduction

In this work we compute the families of classical Hamiltonians with two degrees of freedom

$$H = \frac{y_1^2 + y_2^2}{2} + V(x_1, x_2)$$

in which the Normal Variational Equation around an invariant plane falls in Shrödinger type with polynomial or trigonometrical potential. In the first case we analyze the integrability of Normal Variational Equation in Liouvillian sense using the Kovacic's algorithm. We also introduce a method of algebrization that transforms equations with transcendental coefficients in equations with rational coefficients without changing the Galoisian structure of the equation. It allow us to deal with the second case via the universal covering of the cylinder. In both cases we obtain Galoisian obstructions to existence of a rational first integral of the original Hamiltonian via Morales-Ramis theory.

Morales-Ramis theory

This is a framework to obtain group-theoretic obstructions to the integrability of hamiltonian systems. The main philosophy consists in the following heuristic: if a Hamiltonian is integrable, then its linear aproximation along any particular solution (in our case the Normal Variational Equation) must be also integrable.

It relates the integrability of Hamiltonian systems, in the frame of Liouville-Arnold theorem, with integrability of linear differential equations in the frame of Picard-Vessiot theory.

In our case, we find obstructions to the existence of rational first integrals. We apply the following result **[1]**.

Theorem. Let H be a Hamiltonian and S a particular solution such that its NVE has irregular singularity at infinity. If H is completely integrable by rational functions, then the identity component of the Galois group of the NVE is abelian.

Our Results

In **[2]** is computed the family of classical Hamiltonians of two degree of freedom, with an invariant plane, such that the NVE along any solution lying on the invariant plane is a Lamé equation. We compute classical Hamiltonians with invariant plane such that the NVE falls in some specific families of Linear differential equations. In particular we find Hamiltonians that have NVE of type *quantum harmonic oscillator equation, Shrödinger equation with polynomial potential of odd degree*, and *Mathieu equation.*

The integrability of those Linear Differential equations is studied by computing their Galois group, using Kovacic's Algorithm. In the case of *Mathieu equation*, where coefficients are not rational, it can be transformed into algebraic form using a change of variables that does not change the Identitiy component of the Galois Group. Then, we find that, in the generic case, the Galois group of all these equations is SL(2,**C**), a connected and non-abelian group.

Then, the following families of Hamiltonians are not integrable using rational functions.

QUANTUM HARMONIC OSCILATOR EQUATION

$$\frac{d^2\psi}{dt^2} = (c_0 t^2 - E)\psi \qquad H = \frac{y_1^2 + y_2^2}{2} + \frac{\lambda_4}{(\lambda_2 + 2\lambda_3 x_1)^2} + \lambda_0 + \lambda_1 x_2^2 + \lambda_2 x_1 x_2^2 + \lambda_3 x_1^2 x_2^2 + \beta(x_1, x_2) x_2^3, \qquad \lambda_3 \neq 0.$$

SHRÖDINGER EQUATION WITH POLYNOMIAL POTENTIAL OF ODD DEGREE

$$\frac{d^2\psi}{dt^2} = (Q_{2m+1}(t) - E)\psi \qquad H = \frac{y_1^2 + y_2^2}{2} + \lambda_0 + P_{2m+1}(x_1) x_2^2 + \beta(x_1, x_2) x_2^3, \qquad \deg(P_{2m+1}(x_1)) = 2m+1.$$

MATHIEU EQUATION

$$\frac{d^2\psi}{dt^2} = (c_0 \cos(t - t_0) - E)\psi \qquad H = \frac{y_1^2 + y_2^2}{2} + \lambda_0 + \begin{cases} \lambda_1 x_1 + \frac{x_1^2}{2} + \lambda_2 x_2^2 + \lambda_3 x_1 x_2^2 + \beta(x_1, x_2) x_2^3, & \lambda_3 \neq 0. \\ \frac{\lambda_1}{(\lambda_3 + 2\lambda_4 x_1)^3} + \frac{\lambda_3 x_1}{8\lambda_4} + \frac{x_1^2}{8} + \lambda_2 x_2^2 + \lambda_3 x_1 x_2^2 + \lambda_4 x_1^2 x_2^2 + \beta(x_1, x_2) x_2^2, & \lambda_4 \neq 0. \end{cases}$$

References

[1] Morales-Ruiz Juan J., Ramis, Jean Pierre. **Galoisian obstructions to integrability of Hamiltonian systems. I, II.** *Methods Appl. Anal.* **8** (2001), no. 1, 33--95, 97--111.

[2] Morales-Ruiz, Juan J., Simó, Carles. **Non-integrability criteria for Hamiltonians in the case of Lamé normal variational equations**. *J. Differential Equations* **129** (1996), no. 1, 111--135.

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