Galois Theory and Spectral Theory
Preliminary Report (Work in progress)

Primitivo B. Acosta-Humánez

joint work with

Juan Morales-Ruiz

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Motivation: Known shape invariant potentials in Quantum Mechanics

Potential	**Name**
$\frac{1}{2}m\omega^2\left(x-\sqrt{\frac{2}{m}}\frac{b}{\omega}\right)^2$	Shifted H. O.
$\frac{1}{2}m\omega^2r^2+\frac{l(l+1)\hbar^2}{2mr^2}-\left(l+\frac{3}{2}\right)\hbar\omega$	3D H.O.
$-\frac{e^2}{r}+\frac{l(l+1)\hbar^2}{2mr^2}-\frac{me^4}{2(l+1)^2\hbar^2}$	Coulomb
$A^2+B^2e^{-2ax}-2B\left(A+\frac{a\hbar}{2\sqrt{2m}}\right)e^{-ax}$	Morse 1
$A^2+\frac{B^2-A^2-\frac{Aa\hbar}{\sqrt{2m}}}{\cosh^2 ax}+\frac{B\left(2A+\frac{a\hbar}{\sqrt{2m}}\right)\sinh ax}{\cosh^2 ax}$	Morse 2

$A^2 + \frac{B^2}{A^2} + 2B\tanh ax - A\frac{A+\frac{a\hbar}{\sqrt{2m}}}{\cosh^2 ax}$	Rosen-Morse 1
$A^2 + \frac{B^2+A^2+\frac{Aa\hbar}{\sqrt{2m}}}{\sinh^2 ar} - \frac{B\left(2A+\frac{a\hbar}{\sqrt{2m}}\right)\cosh ar}{\sinh^2 ar}$	Rosen-Morse 2
$A^2 + \frac{B^2}{A^2} - 2B\coth ar + A\frac{A-\frac{a\hbar}{\sqrt{2m}}}{\sinh^2 ar}$	Eckart 1
$-A^2 + \frac{B^2+A^2-\frac{Aa\hbar}{\sqrt{2m}}}{\sin^2 ax} - \frac{B\left(2A-\frac{a\hbar}{\sqrt{2m}}\right)\cos ax}{\sin^2 ax}$	Eckart 2
$-(A+B)^2 + \frac{A\left(A-\frac{a\hbar}{\sqrt{2m}}\right)}{\cos^2 ax} + \frac{B\left(B-\frac{a\hbar}{\sqrt{2m}}\right)}{\sin^2 ax}$	Pöschl-Teller 1
$(A-B)^2 - \frac{A\left(A+\frac{a\hbar}{\sqrt{2m}}\right)}{\cosh^2 ar} + \frac{B\left(B-\frac{a\hbar}{\sqrt{2m}}\right)}{\sinh^2 ar}$	Pöschl-Teller 2

(R. Dutt, A. Khare, U.P. Sukhatme, Am.J.Phys. 56 (1988) 163-168)

Preliminaries and notations

The One-dimensional Stationary Schrödinger Equation (SSE) is given by

$$S_{(u,x)}\Psi = \lambda\Psi, \quad S_{(u,x)} = \frac{d^2}{dx^2} - u(x). \tag{1}$$

In quantum mechanics x is the **cartesian or radial coordinate** , Ψ is the **wave function**, the eigenvalue λ is the **energy** level, $u(x)$ is the **potential or potential energy** and the solutions of (1) are the **eigenfunctions** of the particle.

Notation. Denote by $\Lambda \subseteq \mathbb{C}$ the set of eigenvalues λ such that (1) is Picard-Vessiot integrable.

Galois groups of SSE

Denotes by Card(Λ) the cardinality of Λ and by $\mathbf{G}_\lambda = \mathbf{G}_\lambda(1)$ the Galois group (over $\mathbf{K}$) of SSE (1) for λ.

Remark. We will see that Λ can be $\emptyset$, i.e., $\mathbf{G}_\lambda = SL(2,\mathbb{C})$ $\forall \lambda \in \mathbb{C}$. On the other hand, if $\lambda_0 \in \Lambda$ then $\mathbf{G}^0_{\lambda_0} \subseteq \mathbb{C}^* \ltimes \mathbb{C}^+$.

Examples. Given $\mathbf{K}$ defined as follows (see [1])

1. $\mathbf{K} = \mathbb{C}(x)$, if $u(x) = P_{2n+1}(x)$, then $\Lambda = \emptyset$.
2. $\mathbf{K} = \mathbb{C}(x)$, if $u(x) = P_{2n}(x)$, $n > 1$, then either, $\Lambda = \{\lambda_0\}$, with $\mathbf{G}_{\lambda_0} = \mathbb{C}^* \ltimes \mathbb{C}^+$ or $\Lambda = \emptyset$.
3. $\mathbf{K} = \mathbb{C}(e^{ix})$, if $u(x) = b\sin(x)$, $b \in \mathbb{C}^*$ then $\Lambda = \emptyset$.

Proposition 1. Given SSE (1), then only one of the following cases can occur for Λ

- $\Lambda = \emptyset$
- $\mathrm{Card}\Lambda = 1$
- Λ is an infinite discrete set
- $\Lambda = \mathbb{C}$ (In this case we say that SSE is "integrable")

Proposition 2. Given SSE (1) with $u(x) \in \mathbb{C}(z(x), z'(x))$. Let be $u(x) = g(z(x))$, $z = z(x)$, and $(z')^2 = \alpha(z)$, we can obtain $\alpha(z)S_{(v,z)}\Psi = \lambda\Psi$, with $v(z) \in \mathbb{C}(z)$ iff

$$v(z) = \frac{\alpha''(z)}{4\alpha(z)} - \frac{3}{8}\left(\frac{\alpha'(z)}{\alpha(z)}\right)^2 + \frac{g(z)}{\alpha(z)}, \quad \frac{\alpha'(z)}{\alpha(z)}, \frac{g(z)}{\alpha(z)} \in \mathbb{C}(z).$$

Remark. In Proposition 2 we present an improvement of the so-called *algebrization*, for details see §2 in [1].

Proposition 3. Let be $b \in \mathbb{C}^*$, $c \in \mathbb{C}$, and $\mathbf{K} = \mathbb{C}(x)$. If $\mathrm{Card}(\Lambda) > 1$ and $u(x) = a^2b^2(bx+c)^2 + \frac{n(n+1)b^2}{(bx+c)^2}$ then only one of the following cases can occur

- $\Lambda = \mathbb{C}$, $\mathbf{G}_\lambda = \mathbf{1}$ or $\mathbf{G}_\lambda = \mathbb{C}^*$, for $a = 0$, $n \in \mathbb{N}$.
- $\Lambda = ab^2(2\mathbb{Z}+1)$, $\mathbf{G}_\lambda = \mathbb{C}^* \ltimes \mathbb{C}^+$, for $a \in \mathbb{C}^*$, $n = 0$.
- $\Lambda = b^2(2\mathbb{Z}+1)$, $\mathbf{G}_\lambda = \mathbb{C}^* \ltimes \mathbb{C}^+$, for $a = \pm 1$, $n \in \mathbb{N}^*$.

Remark. Taking $a = 0$ and $\lambda = 0$ in Proposition 3 we obtain

$$\Psi_\pm = (bx + c)^{\frac{1 \pm (1+2n)}{2}}, \quad \mathbf{G}_0 = \mathbf{1}.$$

Proposition 4. Let be $\mathbf{K} = \mathbb{C}(x)$. If there exists λ_0 such that the Galois group of SSE (1) is either tetrahedral, octahedral or icosahedral group, then $\mathbf{G}_\lambda = SL(2, \mathbb{C})$ for $\lambda \in \mathbb{C} - \{\lambda_0\}$.

Remark This proposition means that for $\mathbf{K} = \mathbb{C}(x)$, $Card(\Lambda) > 1$ implies that SSE (1) never falls in case 3 of Kovacic algorithm.

Darboux Transformation (DT)

Theorem (Darboux). Let be $\Lambda \neq \emptyset$, suppose that $\Psi_\lambda = \Psi_\lambda(x)$ is an eigenfunction of SSE (1) for all $\lambda \in \Lambda$, $\Psi_{\lambda_1} = \Psi_{\lambda_1}(x)$ an eigenfunction for $\lambda_1 \in \Lambda$. Then for the same Λ we can obtain the SSE

$$S_{(v,x)}\widetilde{\Psi} = \lambda\widetilde{\Psi}, \quad v(x) = \Psi_{\lambda_1}\left(\frac{1}{\Psi_{\lambda_1}}\right)'' - \lambda_1. \qquad (2)$$

Furthermore, if $\lambda \neq \lambda_1$ then $\widetilde{\Psi}_\lambda = \widetilde{\Psi}_\lambda(x)$ given by

$$\widetilde{\Psi}_\lambda = \Psi'_\lambda - \frac{\Psi'_{\lambda_1}}{\Psi_{\lambda_1}}\Psi_\lambda = \frac{W(\Psi_{\lambda_1}, \Psi_\lambda)}{W(\Psi_{\lambda_1})}$$

is an eigenfunction of SSE (2) for all $\lambda \in \Lambda$.

Theorem (Darboux iteration). Let be $\Lambda \neq \emptyset$, SSE

$$S_{(u_n,x)}\Psi_n = \lambda\Psi_n, \quad u_n = u_n(x) \in \mathbf{K}_n. \tag{3}$$

Let assume that $f_n = f_n(x, \lambda_n)$ is an eigenfunction for $\lambda = \lambda_n$ and $\Psi_{(n,\lambda)}$ is an eigenfunction for all $\lambda \in \Lambda$ of SSE (3), then for the same Λ we can obtain the SSE

$$\begin{gathered} S_{(u_{n+1},x)}\Psi_{n+1} = \lambda\Psi_{n+1}, \quad u_{n+1} \in \mathbf{K}_{n+1}, \\ u_{n+1} = u_n - 2\left(\ln f_n\right)'' = u_0 - 2\sum_{k=0}^{n}\left(\ln f_k\right)''. \end{gathered} \tag{4}$$

Furthermore, if $\lambda \neq \lambda_n$ then $\Psi_{(n+1,\lambda)}$ given by

$$\Psi_{(n+1,\lambda)} = \Psi'_{(n,\lambda)} - \Psi_{(n,\lambda)}\frac{f'_n}{f_n}$$

is an eigenfunction of SSE (4).

Theorem (Crum iteration) Let be Λ with $Card(\Lambda) > 1$, suppose that $\Psi_\lambda = \Psi_\lambda(x)$ is an eigenfunction of SSE (1) for all $\lambda \in \Lambda$ and $\Psi_{\lambda_1}, \ldots, \Psi_{\lambda_n}$ are eigenfunctions of the SSE (1) for $\lambda_1 \neq \ldots \neq \lambda_n \in \Lambda$. Then for the same Λ we can obtain SSE

$$
\begin{aligned}
&S_{(u[n],x)}\Psi[n] = \lambda\Psi[n], \quad u[n] \in \mathbf{K}_n, \\
&u[n] = u - 2\left(\ln W(\Psi_{\lambda_1}, \ldots, \Psi_{\lambda_n})\right)''
\end{aligned}
\tag{5}
$$

where $\Psi_\lambda[n] = \Psi_\lambda[n](x)$ given by

$$
\Psi_\lambda[n] = \frac{W(\Psi_1, \ldots, \Psi_n, \Psi_\lambda)}{W(\Psi_1, \ldots, \Psi_n)}.
$$

is an eigenfunction of SSE (5) for all $\lambda \in \Lambda$.

Proposition 5. Let be f_n as in Darboux iteration, $\Psi_{(\lambda_n)}$ as in Crum iteration. If $(\ln f_n)'$ and $(\ln \Psi_{(\lambda_n)})'$ are algebraic over $\mathbf{K}_n$, then

1. $\mathbf{G}_\lambda(3) = \mathbf{G}_\lambda(4), \quad \mathbf{K}_n = \mathbf{K}_{n+1}$,
2. $(\mathbf{G}_\lambda(3))^0 = (\mathbf{G}_\lambda(4))^0, \quad \mathbf{K}_n \neq \mathbf{K}_{n+1}$.

Remark.(Iteration over the same eigenvalue) Let be $\lambda_0 \in \Lambda$, we can see that is not possible to get the solution $\Psi_{(n+1,\lambda_0)}$ through $\Psi_{(n+1,\lambda_0)}$ using Darboux theorem, for instance we use Kovacic's algorithm. In this case $\mathbf{G}_\lambda(3) = \mathbf{G}_\lambda(4), \quad \mathbf{K}_n = \mathbf{K}_{n+1}$. Finally, we can see that $Card(\Lambda) = 1$ if and only if $u_{n+2} = u_n$. (Test)

Examples

1. Starting with $u_0 = 0$ we can see that $\Lambda = \mathbb{C}$, the eigenfunctions are given by $\Psi_{(0,0)} = c_1 + c_2 x$, and for $\lambda \neq 0$, $\Psi_{(0,\lambda)} = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$, now taking $\lambda_0 = 0$ and $f_0 = x$, we obtain $u_1(x) = \frac{2}{x^2}$, $\Psi_{(1,0)} = \frac{c_1}{x} + c_2 x^2$, and for $\lambda \neq 0$,

$$\Psi_{(1,\lambda)} = \frac{c_1(-\sqrt{\lambda} + \lambda x)e^{\sqrt{\lambda}x}}{x} + \frac{c_2(\lambda x + \sqrt{\lambda})e^{-\sqrt{\lambda}x}}{x}.$$

In this way we have that $\mathbf{G}_0(3) = \mathbf{G}_0(4) = \mathbf{1}$, and for $\lambda \neq 0$ we have $\mathbf{G}_\lambda(3) = \mathbf{G}_\lambda(4) = \mathbb{C}^*$, in both cases $\mathbf{K_0} = \mathbf{K_1} = \mathbb{C}(x)$. We see that for $\lambda_n = \lambda = 0$, $\Psi_{(n,0)} = \frac{c_1}{x^n} + c_2 x^{n+1}$, $u_n = \frac{n(n+1)}{x^2}$, $\mathbf{G}_0(3) = \mathbf{G}_0(4) = \mathbf{1}$ and $\mathbf{K}_n = \mathbb{C}(x)$.

We can see that for $\lambda_0 = -1$ we take $f_0 = \cos x$. In this way we obtain $u_1 = 2 + 2\tan^2 x = \frac{2}{\cos^2 x} \in \mathbf{K_1}$, where $\mathbf{K}_1 = \mathbb{C}(\sin x, \cos x)$. Now, if instead of $f_0 = \cos x$ we take $f_0 = e^{ix}$, then $u_1 = 1$. In the same way, we can take $\lambda_0 = 1$ so that f_0 can be e^x or $\cosh(x)$. In general, if $u_0(x) = 0$, $f_n(x) = g_n(x)e^{\sqrt{\lambda_n}x}$, where $g_n(x) \in \mathbb{C}(x)$, then $\mathbf{K}_n = \mathbb{C}(x)$, and $\mathbf{G}_\lambda(3) = \mathbf{G}_\lambda(4) = \mathbb{C}^*$ for $\lambda \neq 0$.

Remark The following potentials can be obtained using Darboux iteration through $u_0 = 0$.

$$u_n = \frac{n(n+1)b^2}{(bx+c)^2}, \quad u_n = \frac{m^2 n(n+1)(b^2-a^2)}{(a\cosh(mx) + b\sinh(mx))^2},$$

$$u_n = \frac{-4abm^2 n(n+1)}{(ae^{mx} + be^{-mx})^2}, \quad u_n = \frac{m^2 n(n+1)(b^2+a^2)}{(a\cos(mx) + b\sin(mx))^2}.$$

By Crum iteration for $\lambda_1 = 1, \lambda_2 = 4$, $\Psi_1 = \cosh x, \Psi_2 = \cosh 4x$, $c_1 = 1, c_2 = 0$, we obtain

$$u[2] = -\frac{24e^{2x}(e^{8x} - 4e^{6x} - 6e^{4x} - 4e^{2x} + 1)}{(e^{6x} + 3e^{4x} - 3e^{2x} - 1)^2}$$

2. Starting with $u_0 = \frac{x^2}{2}$ we can see that $\Lambda = 2\mathbb{Z} + 1$, so that $\lambda = 2m + 1$. In this case we obtain the classical solutions for harmonic oscillator through Hermite Polynomials. Darboux iteration an Crum iteration produces the same result. We can see that $\mathbf{G}_\lambda = \mathbb{C}^* \ltimes \mathbb{C}$ and $\mathbf{K}_n = \mathbb{C}(x)$ for equations (1), (3), (4) and (5).

Remark. The potential $u_n = x^2 + \frac{n(n+1)}{x^2}$ can be obtained trough Darboux iteration. The Galoisian structure is the same of the harmonic oscillator.

Relationship with Morales-Ramis theorem

Theorem (Morales - Ramis, 1998). *If hamiltonian H is completely integrable in the Liouville sense then the Identity component of Galois Group of the* **Normal Variational Equation** *is abelian.*

Proposition 6. SSE (1) is integrable if and only if $\mathbf{G}_\lambda^0 \subseteq \mathbb{C}^*$ for all λ.

In particular, Morales - Ramis version for SSE is:

If SSE (1) is integrable, then the Identity component of Galois Group of SSE (1) for $\lambda = \lambda_0$ is abelian.

Projects and future work

1. To clarify the relationship between Parametric Picard - Vessiot theory and Quantum mechanics.
2. Obtain a Galoisian classification of the shape invariant and other quantum mechanics solvable potentials. Try to contribute with new potentials of these types
3. To clarify the relationship between Interwining operators (a generalization of Darboux transformations) and Eigenrings.
4. Applications to solve specials partial differential equations such as KdV, KP and non-stationary SE.

References

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Thank you

For comments and suggestions please contact us in

DEPARTAMENTO DE MATEMÁTICA APLICADA II
UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONA, SPAIN
e-mail: primitivo.acosta@upc.edu
e-mail: juan.morales-ruiz@upc.edu