# On groups with $L$-presentations

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# Some languages associated to groups

$G = \langle S \rangle$, $S = S^{-1}$, $S$ finite

- Word Problem $= \{\ w \in S^* \mid w =_G 1\ \}$
- Geodesics $= \{\ w \in S^* \mid w =_G u \Rightarrow |w| \leq |u|\ \}$

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For $g \in G$,

$$\ell(g) = \min\{|w| \mid w \in S^*,\ g =_G w\ \}$$

- Co-Word Problem $= \{\ w \in S^* \mid w \neq_G 1\ \}$
- Dead ends $=$
  $\{\ w \in S^* \mid \ell(w) = |w|,\ \ell(ws) \leq \ell(w),\ \text{for } s \in S\ \}$
  - ...

# Remarks on dead ends

**Theorem (2006,Š)**

*For every finitely generated infinite group* $G = \langle S \rangle$, *there exists a generating set* $S'$ *(with* $|S'| \leq 4|S| + 2$*) with respect to which the language of dead ends in non-empty.*

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**Conjecture (Elder, Gutierrez, Š)**

*The language of geodesics of the first Grigorchuk group $\mathcal{G} = \langle a, b, c, d \rangle$ is precisely the set of prefixes of the language of dead ends.*

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**Theorem (2006, Holt, Röver)**

*The co-word problem of $\mathcal{G}$ is indexed.*

# What kinds of languages?

### Theorem (1985, Lysënok)

*A presentation for $\mathcal{G}$ is given by*

$$\mathcal{G} = \langle a, b, c, d \mid \phi^n(R),\ \ n \geq 0 \rangle,$$

*where*

$$R = \{\ a^2, b^2, c^2, d^2, bcd, (ad)^4, (adacac)^4\ \}$$

*and $\phi : S^* \to S^*$ is the endomorphism of $S^*$ defined by the letter substitutions*

$$\phi : \qquad a \mapsto aca, \qquad b \mapsto d, \qquad c \mapsto b, \qquad d \mapsto c.$$

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**Theorem (Elder, Gutierrez, Š)**

*The language of geodesics of $\mathcal{G}$ is closed under iterations of $\phi$.*

# $L$-presentations

## Definition

An $L$-presentation of a group is a quadruple

$$P = \langle\, X \mid Q \mid R \mid \Phi \,\rangle,$$

where

- $X$ is a set of letters
- $Q$ is a set of reduced group words over $X$
- $R$ is a set of reduced group words over $X$
- $\Phi$ is a set of endomorphisms $\phi : F(X) \to F(X)$.

$P$ is finite if $X, Q, R, \Phi$ are finite.
$P$ is ascending if $Q$ is empty.

# $L$-presentations

An $L$-presentation $P = \langle X, Q, R, \Phi \rangle$ defines a group through the ordinary presentation

$$G = G_P = \left\langle \; X \mid Q \cup \left( \bigcup_{\phi \in \Phi^*} \phi(R) \right) \; \right\rangle$$

# Examples

## Example

$$F = \langle\, x_0, x_1, \dots, \mid \emptyset \mid x_1^{x_0} = x_2 \mid \phi, \psi \,\rangle$$

where

$$\phi : \qquad x_i \mapsto x_{i+1}, \text{ for } i \geq 0$$

and

$$\psi : \qquad x_0 \mapsto x_0, \qquad x_i \mapsto x_{i+1}, \text{ for } i \geq 1.$$

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## Example (2003, Bartholdi)

$$L = \langle\ a, b, t \mid a^2, a^{-1}b \mid [a, b] \mid \phi\ \rangle$$

where

$$\phi : \qquad a \mapsto a, \qquad b \mapsto b^t, \qquad t \mapsto t.$$

**Theorem (2003, Bartholdi)**

*If $G$ has a finite ascending $L$-presentation with respect to $S$ then it has a finite ascending $L$-presentation with respect to any finite generating set.*

# ascending versus non-ascending

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**Example**

The lamplighter group $L$ does not have an ascending $L$-presentation with respect to $\{a, t\}$.

Thus the class of groups with finite $L$-presentations and finite ascending $L$-presentations are distinct.

Example

Consider

$$\overline{G} = \langle\ a, b \mid babab = a^2\ \rangle$$

and its normal subgroup

$$G = \langle b \rangle^{\overline{G}} = \{\ g \in \overline{G} \mid g =_{\overline{G}} w,\ \text{for some word } w,\ \exp_a(w) = 0\ \}.$$

Then

$$G = \langle\ \dots, b_{-1}, b_0, b_1, b_2, \cdots \mid b_2 b_1 b_0 \mid \phi\ \rangle$$

where

$$\phi : \qquad b_i \mapsto b_{i+1},\ \text{for all } i.$$

# a non-co-hopfian f.g. subgroup in a hyperbolic group

Theorem (2001, Kapovich, Wise)

*There exists a hyperbolic (in fact $C'(1/6)$) group*

$$\overline{G} = \langle\ x, y, t \mid r(x, y),\ x^t = \phi(x),\ y^t = \phi(y)\ \rangle$$

*such that*

$$G = \langle\ x, y \mid r(x, y) \mid \phi\ \rangle$$

*is non-hyperbolic, non-hopfian, one-ended, not finitely presented, not quasi-convex in $\overline{G}$, etc.*

Recall that

$$\mathcal{G} = \langle a, b, c, d \mid \phi^n(R),\ \ n \geq 0 \rangle,$$

where

$$R = \{\ a^2, b^2, c^2, d^2, bcd, (ad)^4, (adacac)^4\ \}$$

and $\phi : S^* \to S^*$ is the endomorphism of $S^*$ defined by the letter substitutions

$$\phi : \qquad a \mapsto aca, \qquad b \mapsto d, \qquad c \mapsto b, \qquad d \mapsto c.$$

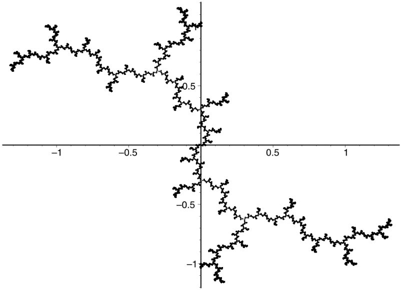
**Theorem (1996, Grigorchuk)**

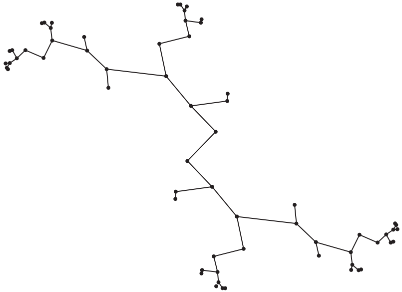
*The group*

$$\overline{G} = \langle\ a, b, c, d, t \mid R,\ \ a^t = aca,\ \ b^t = d, c^t = d,\ \ d^t = c\ \rangle$$

*is finitely presented, amenable but not elementary amenable group.*

# Iterated monodromy group $IMG(z^2 + i)$



Theorem (2006, Grigorchuk, Savchuk, Š)

*An $L$-presentation for $G = IMG(z^2+i)$ is given by*

$$\langle a, b, c \mid R \mid \phi \rangle$$

*where*

$$R = \{\ a^2, (ac)^4, [c, ab]^2, [c, bab]^2, [c, ababa]^2, [c, ababab]^2, [c, bababab]^2\ \}$$

*and*

$$\phi: \qquad a \mapsto b, \qquad b \mapsto c, \qquad , c \mapsto aba.$$

# $L$-presentations for $IMG(z^2 + c)$

Recently, Bartholdi and Nekrashevych have provided $L$-presentations for all iterated monodromy groups of post-critically finite quadratic polynomials.