The automata that define representations of monomial algebras

Sarah Rees,
University of Newcastle, UK

Hoboken, April 2007

The representations of string algebras (particular quotients of path algebras) were studied and classified by Butler and Ringel (1987).

Two types of representations: **strings** and **bands**.

The string representations correspond to a set S of strings over a finite alphabet, which is regular.

The bands fall into parameterised families, each family defined by an element of a subset B of S.

In fact
- S is locally testable, hence star-free
- the minimal automaton is easily constructible,
- B and its properties are easily identifiable from the automaton.

This all works not just for string algebras but actually for any monomial algebra $A = kQ/\langle P \rangle$; except that in the more general situation A has further classes of representations.

My aim: to explain where these representations come from and how a little automata theory, mixed with geometry/topology, facilitates their study.

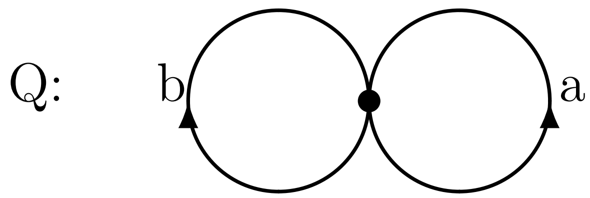
Path, monomial and string algebras

Q is a finite directed graph, called the **Gabriel quiver**, vertices Q_0, directed edges Q_1 (called **arrows**), P is a set of paths in Q, k a field (algebraically closed).

The **path algebra** kQ is the k-algebra spanned by directed paths in Q, the product of 2 paths being their concatenation (read left to right).

The **monomial algebra** $kQ/\langle P\rangle$ is obtained from kQ by setting every path in P to zero. Certain restrictions on the relationship between Q and P make this a **string algebra**.

An example of a string algebra:



$$P = \{a^3, b^3, ab, ba\}.$$

The representations of $A = kQ/\langle P \rangle$.

A module for A can be described as $\oplus_{v \in Q_0} X_v$, with specified actions $a : X_{s(a)} \to X_{t(a)}$, for each $a \in Q_1$.

Let $\theta : \tilde{Q} \to Q$ be the universal cover of Q.

We can define an indecomposable module for A from any subtree Γ of the tree $\tilde{Q}$ which contains no path covering a path in P. Each $\xi \in \Gamma_0$ defines a basis vector $v_\xi \in X_{\theta(\xi)}$. For $\xi \in \Gamma_0, a \in Q_1$,
if $\exists (\xi, \eta) \in \Gamma_1$ mapped by θ to a, then $v_\xi \mapsto^a v_\eta$,
otherwise $v_\xi \mapsto^a 0$.

For a string algebra, Γ is forced to be an undirected path (no branches).

Such a path (if of length > 0) is determined (up to isomorphism) by a sequence over $Q_1 \cup Q_1^{-1}$ which
- defines an undirected path in Q,
- contains no backtracks,
- and contains no path in $P \cup P^{-1}$ as a subpath.

We call this a string.

The reverse string determines the same Γ.

Suppose that $s = x_1 \dots x_n$ is a string.

As we just described, $s = x_1 \dots x_n$ determines an $n+1$-dim module with basis $e_1, \dots e_{n+1}$, so that
if $x_i = a$, then $e_i \mapsto^a e_{i+1}$,
and if $x_i = a^{-1}$, then $e_{i+1} \mapsto^a e_i$.

Now if
• the image of s in Q is a closed path but not a proper power of one, and
• s^m is always a string, for $m \geq 1$,
then s is also a band.

In that case, for $\nu \in k$, $m \geq 1$,
we define an mn-dimensional module $\oplus_{i=1}^n E_i$,
where $E_1, \dots E_n$ are m-dimensional over k,
and where (wrt selected bases, indices taken mod n)
if $i > 1$ and $x_i = a$, $E_i \mapsto^a E_{i+1}$ as I_m,
if $i > 1$ and $x_i = a^{-1}$, $E_{i+1} \mapsto^a E_i$ as I_m,
if $x_1 = a$, $E_1 \mapsto^a E_2$ as $J_{m\nu}$,
if $x_1 = a^{-1}$, $E_2 \mapsto^a E_1$ as $J_{m\nu}$,
where $J_{m\nu}$ is an $m \times m$ Jordan block matrix with eigenvalue ν.

Since, by definition, strings are sequences over $Q_1 \cup Q_1^{-1}$ avoiding certain subsequences, S is certainly locally testable, hence star-free.

It's straightforward to construct an automaton M' recognising S. The target state of an input sequence $\underline{x} = x_1 \dots x_r$ needs to record both x_r and the maximal suffix of $\underline{x}$ which is a prefix of a string in $P \cup P^{-1}$.
• Seen as a directed graph (with failure states excluded) M' maps via a graph morphism to Q.
• M' also maps via a directed graph morphism to the minimal automaton M.
• The only vertices of M which are the images of more than one vertex in M' are either sinks or sources.

It is clear from the above description of M' that any band must label a circuit in M', and hence also in M. (It could also label some non-closed paths.)

Since S is star-free, no minimal circuit in M is powered. Since also S is substring closed, every minimal circuit in M is labelled by a band.

So

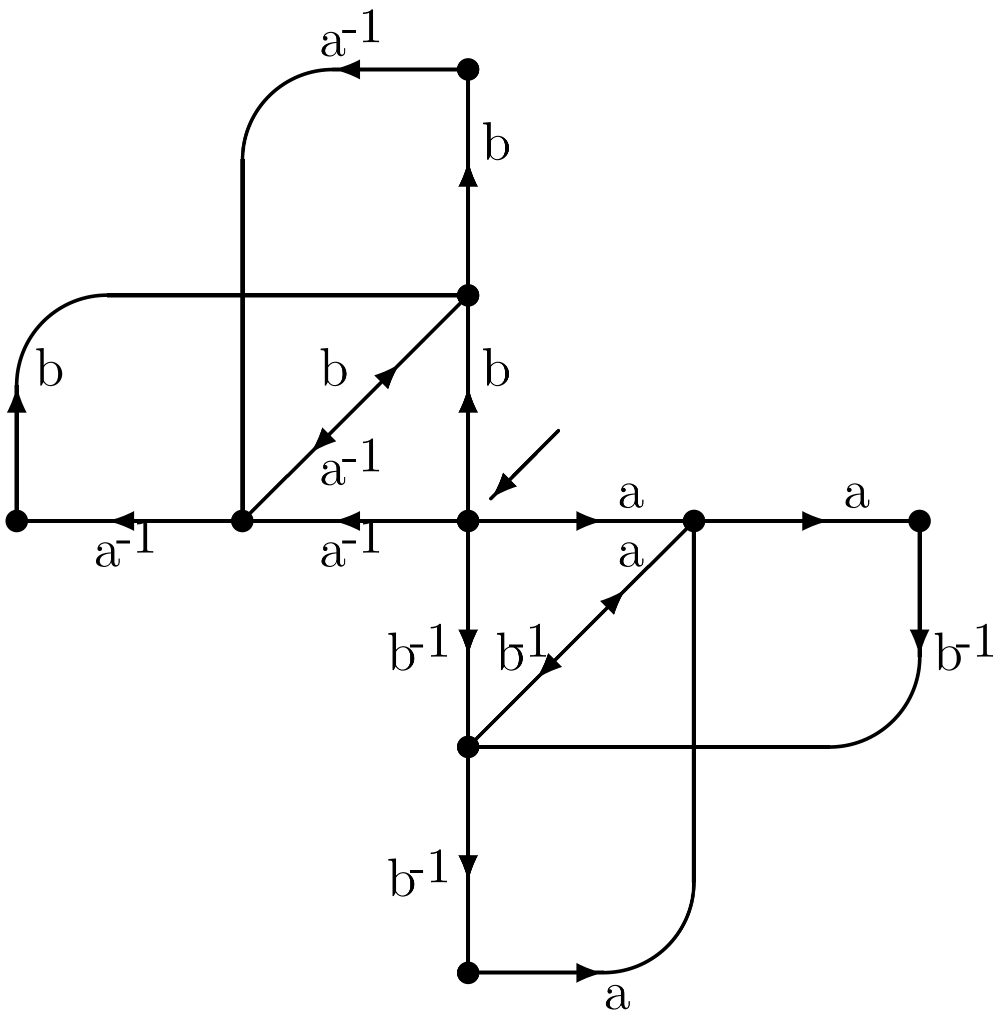
S is the set of labels of all paths in M.

B is the set of labels of all circuits in M which are not powers of circuits.

B is infinite $\iff$ M contains intersecting circuits.

For if γ_1, γ_2 are distinct intersecting minimal circuits, then we can form infinitely many bands of the form $\lambda_1\gamma_1 + \lambda_2\gamma_2$. In that case, $\exists C$ such that for infinitely many n, the number of bands of length n exceeds C^n.

The string/band automaton of our example



In this example we see bands

$$a^2b^{-1}, \quad ab^{-1}, \quad a^{-1}b^2, \quad a^{-2}b$$

of length 3, and longer ones such as those of the form

$$(a^2b^{-1})^i(b^{-1}ab^{-1})^j.$$