The Submonoid
and
Rational Subset Membership Problems
for
Graph Groups

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joint work with

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Some algorithmic problems I

G is a group with finite monoid generating set A.

- **The Word Problem:**
  - Given $w \in A^*$, is $w = 1$ in G?
- **The Generalized Word Problem:**
  - Given $w_1, \ldots, w_n, w \in A^*$, is $w \in \langle w_1, \ldots, w_n \rangle$ in G?
- **The Submonoid Membership Problem:**
  - Given $w_1, \ldots, w_n, w \in A^*$, is $w \in \{w_1, \ldots, w_n\}^*$ in G?
- Submonoid membership problem for $\mathbb{Z}^n$ is the well-known problem of integer programming.
- Notice: $g \in G$ has finite order iff $g^{-1} \in g^*$, so decidable submonoid membership implies computability of the order of an element.

Some algorithmic problems II

- **The Rational Subset Membership Problem:**
  - Given a finite state automaton $\mathscr{A}$ over A and $w \in A^*$, does w belong to the rational subset of G represented by $\mathscr{A}$?
- Since finitely generated submonoids are rational subsets, this problem is more general than the previous ones.
- A theorem of Anisimov and Seifert shows the only subgroups which are rational are the finitely generated ones.
- Double cosets of finitely generated subgroups are rational subsets.
- Solving equations with rational constraints plays a role in the decidability of the positive theory for graph groups.

Double cosets as rational subsets

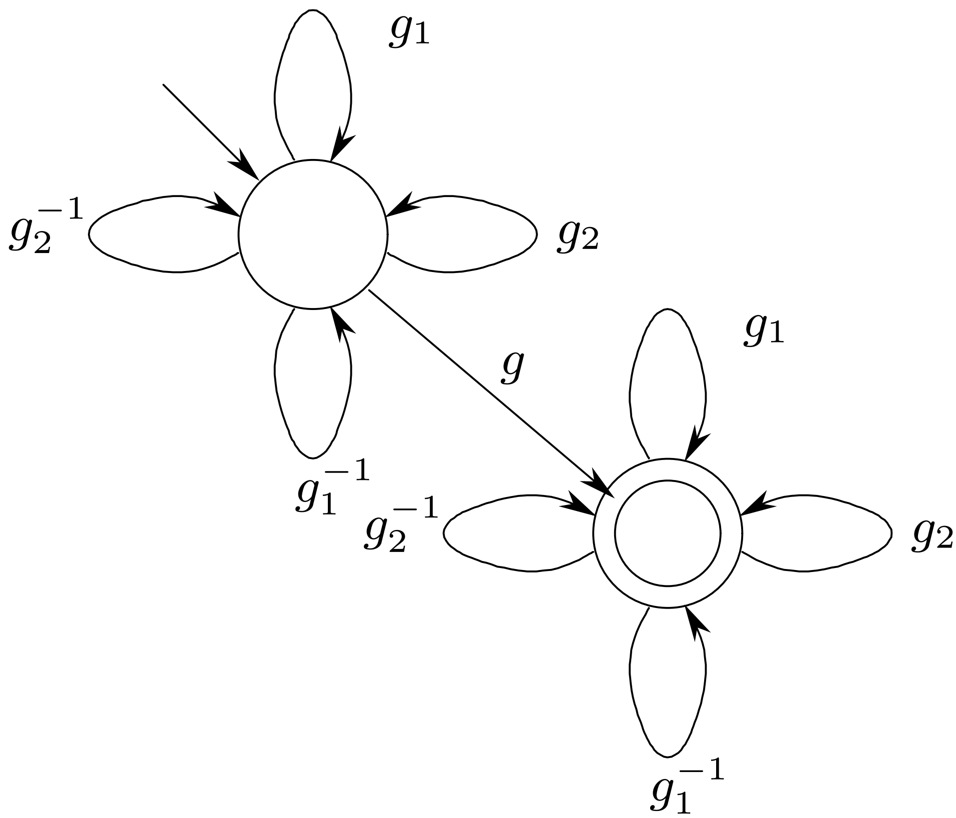


Figure 1: Automaton accepting the double coset $\langle g_1, g_2\rangle g\langle g_1, g_2\rangle$.

Distinguishing the problems

- Mihailova showed $F_2 \times F_2$ has a decidable word problem but undecidable generalized word problem.
- The proof basically amounts to the Chomsky-Schützenberger construction.
- The Rips construction gives hyperbolic examples.
- The goal of this talk is to provide the first example of a group with *decidable* generalized word problem but *undecidable* submonoid membership problem.
- Like $F_2 \times F_2$ it is a "natural" group.
- We also give some indication that the submonoid and rational subset membership problems are more closely linked than one might have guessed.

Graph groups

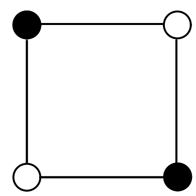
- If $\Gamma = V \cup E$ is a simplicial graph, the associated *graph group* is:

$$\mathscr{G}(\Gamma) = \langle V \mid xy = yx, (x, y) \in E \rangle.$$

- Notice one can define a monoid $\mathscr{M}(\Gamma)$ by the same presentation; such monoids are called *trace monoids* and predate graph groups by about 10 years.
- An edgeless graph gives rise to a free group.
- A complete graph yields a free abelian group.
- A complete bipartite graph gives rise to a direct product of free groups.
- Disjoint union corresponds to free product.
- Simplicial join corresponds to direct product.

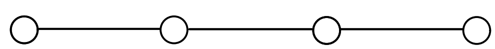
The key players

- A cycle C4 on 4 vertices yielding $F_2 \times F_2$:



  - Kambites proved a graph group contains $F_2 \times F_2$ iff it contains an induced C4.

- A path P4 on 4 vertices:



  - Niblo and Wise proved that $\mathscr{G}(\mathsf{P4})$ is not subgroup separable.

- Metaftsis and Raptis showed a graph group is subgroup separable iff it contains no induced C4 or P4.

Algorithmic problems in graph groups

- Graph groups have decidable word problem (linear time on a RAM machine).
- A result of Kapovich, Myasnikov and Weidmann shows that a graph group has decidable generalized word problem if the graph contains no induced CN with $N \geq 4$.
- Generalized word problem is open for the pentagon $\mathsf{C5}$.
- In particular $\mathscr{G}(\mathsf{P4})$ has decidable generalized word problem.
- Proof is based on a splitting of such groups over free abelian groups.

The main result

Theorem 1 (ML, BS). *Let* Γ *be a simplicial graph. Then the following are equivalent:*

1. $\mathscr{G}(\Gamma)$ *has decidable submonoid membership problem;*
2. $\mathscr{G}(\Gamma)$ *has decidable rational subset membership problem;*
3. Γ *does not contain an induced* $\mathsf{C4}$ *or* $\mathsf{P4}$*.*

- Therefore, the group $\mathscr{G}(\mathsf{P4})$ has decidable generalized word problem but undecidable submonoid membership problem.
- There is a fixed submonoid of $\mathscr{G}(\mathsf{P4})$ with undecidable membership problem.

Proof ideas

- The graph groups with no induced C4 or P4 are precisely the groups built up from the trivial group by taking direct products with $\mathbb{Z}$ and forming free products.
- We define a class of formal languages called SLI languages (Semilinear Intersection).
- This class is closed under rational transduction and so groups with SLI word problem are well defined and closed under taking finitely generated subgroups and finite extensions.
- We prove closure of this class of groups under direct product with $\mathbb{Z}$ and under free product.
- Parikh's theorem plays a key role in the proofs.
- All SLI groups have decidable rational subset problem.

A decidability result

Theorem 2 (ML, BS). *Let $\mathscr{C}$ be the smallest class of groups containing the trivial group closed under:*

1. *Taking finitely generated subgroups;*
2. *Finite extensions;*
3. *Free product;*
4. *Direct product with $\mathbb{Z}$.*

Then every group in $\mathscr{C}$ has decidable generalized word problem.

- This class contains all groups known to have decidable rational subset membership problem to the best of our knowledge.

The undecidability result

- We just need to explain the undecidability of the submonoid membership problem for P4.
- **Rational subset intersection problem:**
  - Given rational subsets K, L of a monoid M, is $K \cap L = \emptyset$?
- For a group, $K \cap L \neq \emptyset$ iff $1 \in KL^{-1}$ and so rational subset intersection is equivalent to rational subset membership.
- Aalbersberg and Hoogeboom showed a trace monoid $\mathscr{M}(\Gamma)$ has decidable rational subset intersection iff Γ has no induced C4 or P4.
- A slight recoding of their result allows us to find a single rational subset of $\mathscr{G}(\mathsf{P4})$ with undecidable membership.
- Let's defer submonoids to later.
- Our decidability result gives an easier proof of the A&H decidability result.

Submonoid membership

G is a free product if $G = H * K$ with $H \neq 1 \neq K$.

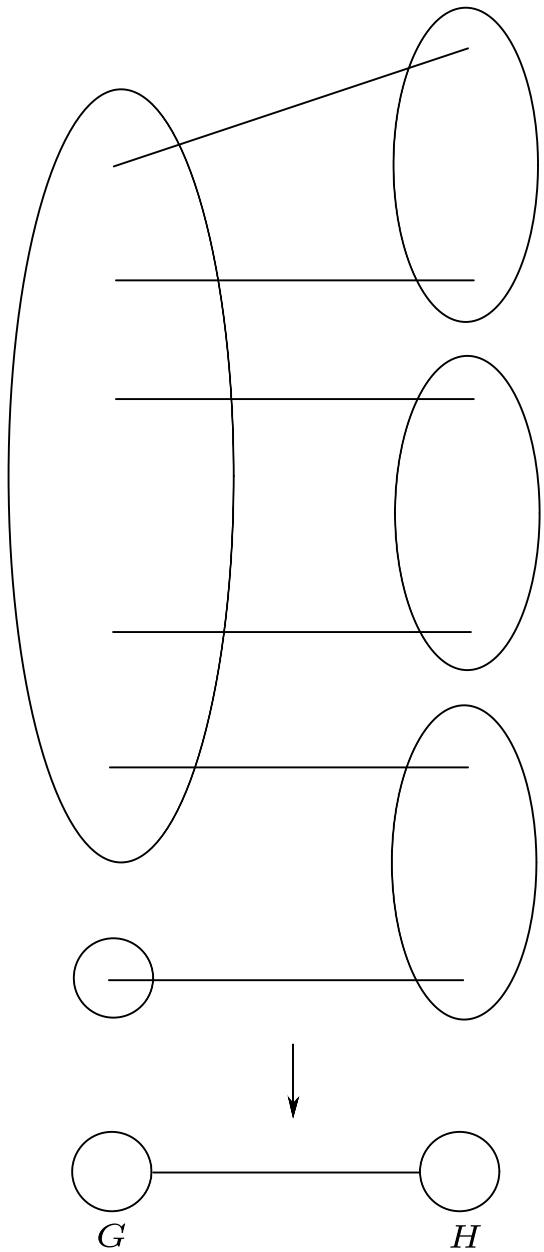
Theorem 3 (ML, BS). *Let G be virtually a free product. Then the rational subset and submonoid membership problems for G are equivalent.*

- A virtually torsion-free or residually finite group with 2 or more ends is virtually a free product by Stallings ends theorem.
- This result gives undecidability of submonoid membership for the graph group associated to P4 together with an isolated vertex.
- It implies exactly one of the following is true:
  - The submonoid and rational subset membership problems are equivalent
  - Decidability of the submonoid membership does not pass through free products.

Proof sketch I

- Decidable rational subset membership is a virtual property (Grunschlag).
- Decidable rational subset membership passes through free products with amalgamation and HNN extensions with finite edge groups (Kambites, Silva, BS).
- So it suffices (by symmetry) to show if $G * H$ has decidable submonoid membership then G has decidable rational subset membership.
- If G is finite, there is nothing to prove so assume G is infinite.
- Then $G * H$ contains $G * F_2$ so we may assume $H = F_2$.

A partial covering



Proof Sketch II

- Let $\mathscr{A}$ be an automaton with:
  - state set Q
  - initial state q_0
  - terminal state $q_f \neq q_0$.
- Let $\tilde{Q} = \{\tilde{q} \mid q \in Q\} \subseteq F_2$ freely generate a free group of rank $|Q|$.
- Let $\Delta = \{\tilde{p} a \tilde{q}^{-1} \mid p \xrightarrow{a} q \in \mathscr{A}\}$
- A simple normal form argument shows g is in the rational subset of G represented by $\mathscr{A}$ iff $\tilde{q}_0 g \tilde{q}_f^{-1} \in \Delta^*$.
- The undecidability of submonoid membership for P4 uses a similar argument, but there is a more technical encoding of states and transitions.