Garside Groups have FFTP

Hoboken, April 2007

Derek Holt

University of Warwick

Group $G = \langle X \rangle$ (X finite), $A = X \cup X^{-1}$.

Let $\mathcal{G}_A$ be the set of geodesic words:
$\mathcal{G}_A = \{ w \in A^* \mid v \in A^*, w =_G v \Rightarrow |w| \leq |v| \}$.

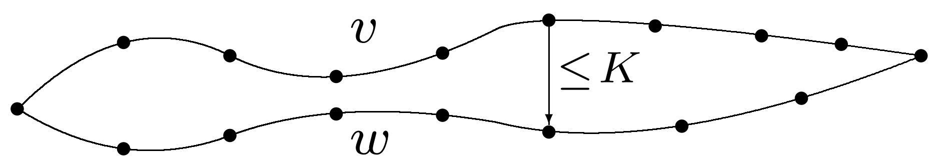
When is $\mathcal{G}_A$ a regular set?

"Conjecture": A group G is automatic if and only if $\mathcal{G}_A$ is regular for some finite monoid generating set A of G.

$\mathcal{G}_A$ is known to be regular for appropriate A for many of the known families of automatic groups including word-hyperbolic groups, virtually abelian groups, geometrically finite hyperbolic groups [Neumann & Shapiro, 95], Coxeter groups [Brink & Howlett, 93], Garside groups [Charney & Meier, 04].

$\mathcal{G}_A$ appears to regular *for all* A only when G is word-hyperbolic or abelian.

Definition The group G has the falsification by fellow traveller property (FFTP) with respect to A if there is a constant $K \geq 0$ such that for any non-geodesic word $v \in A^*$, there exists a word $w \in A^*$ with $v =_G w$ and $|w| < |v|$ such that v and w K-fellow travel.



Conjecture: A group G has FFTP with respect to some finite generating set of G if and only if $\mathcal{G}_A$ is regular for some finite generating set G.

A group with FFTP with respect to A has regular geodesics with respect to A.

Groups with FFTP are finitely presented, they have quadratic Dehn function, and they have a rational growth function.

Example: (Elder, 05)

$$G = \langle\, t, a \mid t^2, [a, tat]\,\rangle,$$

with $A = \{a, a^{-1}, t\}$.

(This is a wreath product of an infinite cyclic group with a group of order 2.)

Then $\mathcal{G}_A$ is regular but G does not have FFTP w.r.t. this generating set.

But it does have FFTP w.r.t. $A = \{a, a^{-1}, b, b^{-1}, t\}$, where $b = tat$.

In fact word-hyperbolic groups, virtually abelian groups, geometrically finite hyperbolic groups, and Coxeter groups have FFTP with respect to suitable generating sets.

Garside Groups

The definition of a Garside group depends on the notion of an *atomic monoid*.

An element m of a monoid M is called *indivisible* or an *atom* if $m \neq 1$ and $m = ab$ implies $a = 1$ or $b = 1$.

M is called *atomic* if it is generated by its indivisible elements and, for each $m \in M$, the supremum of the lengths of words $a_1 a_2 \cdots a_r$ equal to m in M and with each a_i atomic is finite.

We can define a partial order with respect to left divisibility on any atomic monoid by $a \leq b$ if $ac = b$ for some $c \in M$.

A Garside group G is defined to be a group having a submonoid G^+ which is atomic, and which has the following additional properties:

1. Left and right cancellation laws hold in G^+;

2. Any two elements of G^+ have a least common multiple and a greatest common divisor on both the left and the right;

3. There exists an element Δ of G^+ with the property that the sets of left and right divisors of Δ are the same, and they form a finite generating set for G^+ as a monoid and G as a group.

Let X be the set of divisors of Δ and $A = X \cup X^{-1}$.

The principal examples of Garside groups are the Artin groups of finite type (which include the braid groups), in which the atoms are the group generators, and Δ is the element in G^+ corresponding to the longest element in the associated (finite) Coxeter group.

The elements of X correspond to the non-identity elements of the associated Coxeter group so, for example, for the braid group B_n, we have $|X| = n! - 1$.

Specific Examples:

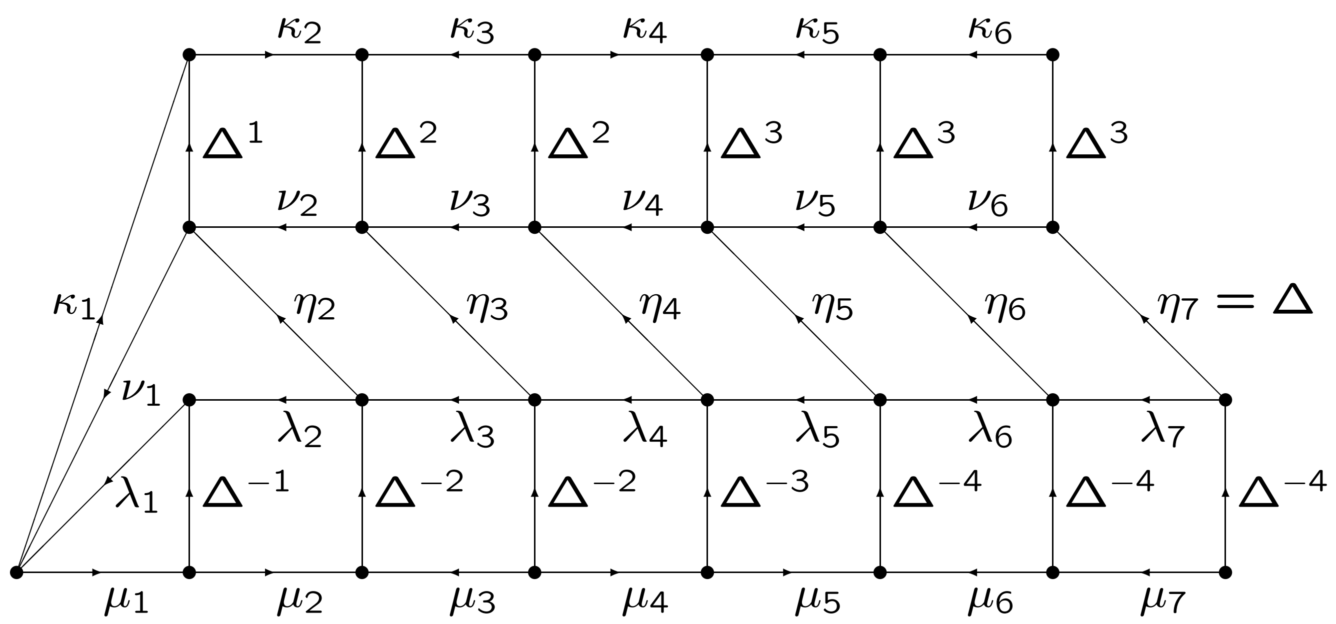
$B_3 = \langle\, a, b \mid bab = aba \,\rangle$, with $\Delta = aba = bab$, $X = \{a, b, ab, ba, aba\}$.

$\langle\, a, b \mid baba = abab \,\rangle$, with $\Delta = abab = baba$, $X = \{a, b, ab, ba, aba, bab, abab\}$.

$B_4 = \langle\, a, b, c \mid bab = aba,\, cbc = bcb,\, ca = ac \,\rangle$, with $\Delta = abacba = \cdots = cbcabc$ (16 expressions).

Charney & Meier [Charney & Meier, 04] proved that $\mathcal{G}_A$ is regular. We can use similar methods to prove

Theorem Any Garside group has FFTP with respect to A. In fact any non-geodesic word 3-fellow travels with a G-equivalent shorter word.



$$w = \mu_1 \mu_2 \mu_3^{-1} \mu_4 \mu_5 \mu_6^{-1} \mu_7^{-1}.$$

$\Delta^4 w^{-1} =_G \lambda_7 \lambda_6 \lambda_5 \lambda_4 \lambda_3 \lambda_2 \lambda_1 =_G \eta_7 \nu_6 \nu_5 \nu_4 \nu_3 \nu_2 \nu_1$, where $\eta_i = \text{gcd}(\Delta, \lambda_i \lambda_{i-1} \cdots \lambda_1)$.

$$w =_G v = \kappa_1 \kappa_2 \kappa_3^{-1} \kappa_4 \kappa_5^{-1} \kappa_6^{-1}.$$

X is closed under conjugation by Δ so w and v 2-fellow travel.