The Cayley Semigroup of a Finite Semigroup

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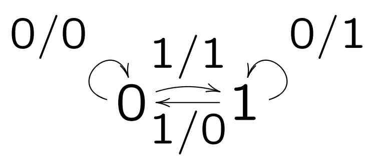
Grigorchuk *et al* started classifying the groups which can be generated by an output automaton.

For $|\Sigma| = |Q| = 2$ they got the following groups: 1, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}$, D_∞, $\mathbb{Z} \wr \mathbb{Z}_2$.

There is something special about the automaton that generates the lamplighter group.



This is the Cayley graph of $\mathbb{Z}_2$, where we output the next state (in Mealy automata notation $\lambda(q, a) = \delta(q, a)$).

Let S be a finite semigroup. We draw a Cayley graph for S by :

S is the set of states.

For every $s \in S$ and $t \in S^1 = S \cup \{1\}$, we add an arrow $s \overset{t/st}{\longrightarrow} st$

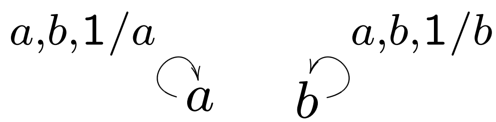
Formally, we may write $\lambda(s,t) = \delta(s,t) = st$

Given a state $s \in S$, we have a function $\varphi_s : (S^1)^* \to S^*$, which can be written as
$\varphi_s([a_1, a_2, \ldots, a_n]) = [sa_1, sa_1a_2, \ldots, sa_1a_2 \cdots a_n]$.

We consider the semigroup $Cayley(S) =< \varphi_s >$

There are 5 semigroups of order 2.

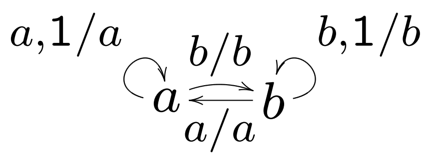
$S_1 =< a, b | a^2 = ab = a, b^2 = ba = b >$, the left zero semigroup of order 2.



$\varphi_a([x_1, x_2, \ldots, x_n]) = [ax_1, ax_1x_2, \ldots, ax_1x_2 \cdots x_n] = [a, a, \ldots, a]$.

We get $Cayley(S_1) \cong S_1$.

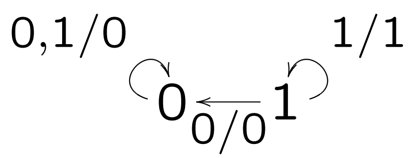
$S_2 =< a, b | a^2 = ba = a, b^2 = ab = b >$, the right zero semigroup of order 2.



$\varphi_a([x_1, x_2, \ldots, x_n]) = [ax_1, ax_1x_2, \ldots, ax_1x_2 \cdots x_n] = [x_1, x_2, \ldots, x_n]$, if $x_1 \neq 1$. $\varphi_a([1]) = [a]$.

We get $Cayley(S_2) \cong S_2$.

$S_3 = (\{0, 1\}, \cdot)$



(S_3 is a monoid so we don't need to add a 1 to the graph)

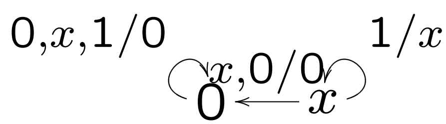
$\varphi_1([x_1, x_2, \ldots, x_n]) = [1x_1, 1x_1x_2, \ldots, 1x_1x_2 \cdots x_n]$

$\varphi_0([x_1, x_2, \ldots, x_n]) = [0, 0, 0, 0, 0]$

clearly $\varphi_0\varphi_1 = \varphi_1\varphi_0 = \varphi_0\varphi_1 = \varphi_0$

We get $Cayley(S_3) \cong S_3$.

$S_4 =< x | x^2 = 0 >$



$$\varphi_x([x_1, x_2, \ldots, x_n]) = [xx_1, xx_1x_2, \ldots, xx_1x_2 \cdots x_n] = [0, 0, \ldots, 0]$$

$$\varphi_x([1]) = [x]$$

$$\varphi_0([x_1, x_2, \ldots, x_n]) = [0, 0, \ldots, 0]$$

We get $Cayley(S_4) \cong S_4$.

$S_5 = Z_2$

We get (Steinberg,Silva) $Cayley(Z_2) = \{a, b\}^+$.

Furthermore, φ_0, φ_1 are invertible. We may consider the group generated by φ_0, φ_1 and obtain the lamplighter group.

Theorem: If S is an idempotent semigroup, then $Cayley(S) \cong S$.

Definition: A finite semigroup is called aperiodic if it has no nontrivial subgroups. equivalently, $s^n = s^{n+1}$ for every $s \in S$ (recall that in general $x^n = x^{n+k}$ in a semigroup).

We extend the result of Steinberg and Silva :

Theorem: If S contains a non trivial subgroup, then S contains a free semigroup on 2 generators.

This motivates us to consider aperodic semigroups.

Rhodes showed that S is aperiodic if and only if $\varphi_s^n = \varphi_s^{n+1}$ for every $s \in S$.

Main Results:

TFAE:

1. S is aperiodic

2. $Cayley(S)$ is finite.

3. $Cayley(S)$ is aperiodic