Associative algebras associated to étale groupoids and inverse semigroups

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Introduction

Groupoid Algebras

Representation Theory

Inverse semigroups

Future work
Background

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- Algebraic properties can often be seen from the groupoid.
- Morita equivalence of groupoid algebras is often explained by a Morita equivalence of the groupoids.
Étale groupoids

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- Counting measure gives a left Haar system in this context.
Discrete analogues

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- Discrete analogues of algebras of higher rank graphs have also been considered.
- Surprising similarities between operator algebras and their discrete analogues have been known for some time.
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- My hopes have since been borne out by J. Brown, L. O. Clark, C. Farthing, A. Sims and M. Tomforde who produce new results faster than I can keep up with.
Groupoid algebras

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• The sum is finite because fibers of \( d \) are discrete and \( f, g \) have compact support.
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• $\mathbb{k}\mathcal{G}$ is unital iff $\mathcal{G}_0$ is compact.
• Leavitt path algebras can be obtained from the usual groupoid for graph $C^*$-algebras (see also later in the talk).
Local bisections

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- The compact local bisections of an ample groupoid $G$ form an inverse semigroup $\Gamma(G)$.
- The map $\Gamma(G) \to \mathbb{k}G$ given by $U \mapsto \chi_U$ is an injective homomorphism.
- It extends to a surjective algebra homomorphism $\mathbb{k}\Gamma(G) \to \mathbb{k}G$.
- When $G$ is Hausdorff, the kernel is generated by $\chi_U + \chi_V - \chi_{U \cup V}$ with $U, V$ disjoint compact open subsets of $G_0$. 
Isotropy

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Let \( G \) be a minimal, Hausdorff ample groupoid.

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Let $\mathcal{G}$ be a minimal, Hausdorff ample groupoid.

1. If $\mathcal{G}_0$ is compact and $\mathcal{G}$ is effective, then $Z(\mathbb{k}\mathcal{G}) = \mathbb{k}1_{\mathcal{G}_0}$.
2. If $\mathcal{G}_0$ is not compact, then $Z(\mathbb{k}\mathcal{G}) = 0$. 
Simplicity

Theorem (L. O. Clark, C. Edie-Michelle)

Let $\mathcal{G}$ be a Hausdorff ample groupoid and $k$ a field. Then $k\mathcal{G}$ is simple if and only if $\mathcal{G}$ is effective and minimal.
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Let $\mathcal{G}$ be a Hausdorff ample groupoid and $\mathbb{k}$ a field. Then $\mathbb{k}\mathcal{G}$ is simple if and only if $\mathcal{G}$ is effective and minimal.

- This was first proved by J.H. Brown, L.O. Clark, C. Farthing and A. Sims over $\mathbb{C}$. 
Morita equivalence

- If $Z$ is a locally compact space and $f: Z \to G_0$ is continuous, there is a pullback groupoid $G[Z]$. 

Theorem (L. O. Clark, A. Sims) If $G$ and $H$ are Morita equivalent, Hausdorff ample groupoids, then $kG$ is Morita equivalent to $kH$. 

Explains why the same moves preserve Morita equivalence between graph $C^*$-algebras and Leavitt path algebras.
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- \( G[Z]_0 = Z, G[Z]_1 = \{ (z, g, z') \mid g: f(z) \to f(z') \} \).
- Groupoids \( G \) and \( H \) are **Morita equivalent** if there is a locally compact space \( Z \) and continuous open surjections \( p: Z \to G_0 \) and \( q: Z \to H_0 \) such that \( G[Z] \cong H[Z] \).
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- It is in fact a $kG$-$kG_x$-bimodule.
- If $f \in kG$ and $t \in L_x$, then

$$f \cdot t = \sum_{d(s) = r(t)} f(s)st.$$

There is an exact functor $\text{Ind}_x : \mathbb{k}G_x\text{-mod} \rightarrow \mathbb{k}G\text{-mod}$ given by

$$M \mapsto \mathbb{k}L_x \otimes_{\mathbb{k}G_x} M.$$
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**Theorem (BS)**

*Let $\mathbb{k}$ be a field and $\mathcal{G}$ an ample groupoid. Then the finite dimensional simple $\mathbb{k}G$-modules are those of the form $\text{Ind}_x(M)$ with the orbit of $x$ finite and $M$ a finite dimensional simple $\mathbb{k}G_x$-module.*
Action on orbits

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- Using ideas from the proof of the simplicity criterion, one can show that $G$ is effective iff $\bigoplus_{x \in G_0} k O_x$ is faithful.
Action on orbits

- $kL_x \otimes_{kG_x} k = kO_x$ where $O_x$ is the orbit of $x$.
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**Theorem (BS, unpublished)**

*Let $G$ be an effective, Hausdorff ample groupoid.*
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1. *If \( k \) semiprimitive, then \( kG \) is semiprimitive.*
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- This is a simple $\mathbb{k}G$-module.
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**Theorem (BS, unpublished)**

*Let* $G$ *be an effective, Hausdorff ample groupoid.*

1. If $\mathbb{k}$ semiprimitive, then $\mathbb{k}G$ is semiprimitive.
2. If $\mathbb{k}$ is a field and $G$ has a dense orbit, then $\mathbb{k}G$ is primitive.
Inverse semigroups

• An inverse semigroup is a semigroup $S$ such that, for all $s \in S$, there exists unique $s^* \in S$ such that

$$ss^*s = s \quad \text{and} \quad s^*ss^* = s^*.$$
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• $E(S)$ is a **semilattice** with $e \wedge f = ef$.
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• The order here is restriction.
• Often we will assume $S$ has a zero element.
Actions of inverse semigroups

- Let $X$ be a Hausdorff topological space with a basis of compact open sets.
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- For group actions, this is the usual action groupoid.
Paterson’s universal groupoid

- Let $\widehat{E(S)} \subseteq \{0, 1\}^{E(S)}$ be the space of non-zero homomorphisms (characters) $\chi : E(S) \to \{0, 1\}$. 
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**Theorem (BS)**

For any commutative ring $k$ with unit, $kS \cong kG(S)$. 
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**Theorem (BS)**

*For any commutative ring $\mathbb{k}$ with unit, $\mathbb{k}S \cong \mathbb{k}\mathcal{G}(S)$.***

- This generalizes Paterson’s result for $C^*$-algebras and my result for finite inverse semigroups.
Tight characters

- Let $S$ be an inverse semigroup with $0$. 
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- Idempotents $e_1, \ldots, e_n \leq e$ cover $e$ if
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- Tight characters were introduced by R. Exel.
- $\chi$ is tight iff $\chi^{-1}(1)$ is a limit of ultrafilters.
Polycyclic monoids

- If $X$ is a set, the polycyclic monoid $P_X$ is the inverse monoid with generators $X$ and relations $x^*x = 1$, $x^*y = 0$ for $x, y \in X$ and $x \neq y$. 
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- $P_X$ is Hausdorff.
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- The space $\widehat{E(S)}_T$ of tight characters is closed and $S$-invariant.
- $\mathcal{G}(S)_T = S \rtimes \widehat{E(S)}_T$ is the tight groupoid of $S$.
- Recall that the idempotents of a commutative ring form a generalized boolean algebra where $e \lor f = e + f - ef$. 

Theorem (BS, unpublished)
Let $S$ be a Hausdorff inverse semigroup. Then $k \mathcal{G}(S)_T$ is isomorphic to $k S/I$ where $I$ is the ideal generated by elements of the form $e - (e_1 \lor \cdots \lor e_n)$ such that $e_1, \ldots, e_n$ cover $e$. 

$\mathcal{G}(P_X)_T$ is the Leavitt path algebra analogue of $O_X$. 

More generally, Leavitt path algebras are the tight algebras of graph inverse semigroups.
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Simple inverse semigroup algebras

- Let $\mathbb{k}$ be a field and $S$ an inverse semigroup.
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- Now \( kS \) can be simple!
- If \( S \) has a non-trivial homomorphimic image, \( kS \) is not simple.
- A semigroup with no non-trivial homomorphimic images is called congruence-free.
Simple inverse semigroup algebras II

- W. D. Munn asked to characterize congruence-free inverse semigroups with simple algebras.
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**Theorem (BS, unpublished)**

Let $S$ be a congruence-free Hausdorff semigroup. Then $kS$ is simple iff no idempotent admits a non-trivial finite cover.
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Let $S$ be a congruence-free Hausdorff semigroup. Then $\mathbb{k}\mathcal{G}(S)_T$ is simple.

• The converse is false.
Effectiveness and minimality

• Let $S$ be a Hausdorff inverse semigroup with zero.
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- Let $S$ be a Hausdorff inverse semigroup with zero.
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Theorem (BS, unpublished)

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2. $G(S)_T$ is minimal if $S$ is 0-simple.
Leavitt path algebras

• Let $G$ be a directed graph and $k$ a field.
Leavitt path algebras

- Let $G$ be a directed graph and $\mathbb{k}$ a field.
- The Leavitt path algebra $L_\mathbb{k}(G)$ is the $\mathbb{k}$-algebra with the same presentation as the graph $C^*$-algebra of $G$. 
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- A graph inverse semigroup is congruence-free iff $G$ is strongly connected and all vertices have in-degree $\geq 2$.
- We recover the result that $G$ strongly connected with no vertex of in-degree 1 implies $L_{\mathbb{k}}(G)$ is simple.
- We also recover semiprimitivity of $L_{\mathbb{k}}(G)$ over a semiprimitive base ring in the case that no vertex has in-degree 1.
Future work

- Characterize primitive and semiprimitive groupoid algebras.
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- Classify all simple modules for groupoid algebras over a field.
- Understand the Jacobson radical of a groupoid algebra.
- Obtain as much as possible of the theory of Leavitt algebras via groupoids.
- Use groupoid algebras to understand cross products.
- Characterize when the tight algebra of an inverse semigroup is simple in semigroup theoretic terms.
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Thank you for your attention!

Obrigado pela sua atenção