Möbius Functions and Semigroup Representation Theory

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Inverse Semigroups

- Just as groups abstract permutations, inverse semigroups abstract partial permutations.
- A semigroup $S$ is an inverse semigroup if, for all $s \in S$, there exists unique $s' \in S$ such that $ss's = s$ and $s'ss' = s'$.
- One writes $s^{-1}$ for $s'$.
- The motivating example is the symmetric inverse monoid $I_n$ of all partial permutations of an $n$-element set.
- The Preston-Wagner theorem says every inverse semigroup of order $n$ embeds in $I_n$. 
A typical element of $I_4$ is

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & 1 & - & 3 \end{pmatrix}.$$ 

Of course

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & - & 4 & - \end{pmatrix}.$$ 

Alternatively, $\sigma$ can be represented by the rook matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

The monoid $R_n$ of all $n \times n$ rook matrices, called the rook monoid, is isomorphic to $I_n$. 

The inverse in this context is the transpose matrix.
Examples

- The signed symmetric inverse monoid consists of all rook matrices with entries in \( \{ \pm 1 \} \).
- It can be identified with the wreath product \( \mathbb{Z}/2\mathbb{Z} \wr I_n \).
- The monoid of uniform block permutations \( UB_n \) consists of all bijections between partitions of \( n \) preserving sizes of blocks.
- A typical example from \( UB_5 \) is

\[
\sigma = \left( \begin{array}{ccc}
\{1, 3\} & \{2\} & \{4, 5\} \\
\{1, 5\} & \{3\} & \{2, 4\}
\end{array} \right)
\]

- \( UB_n \) can be identified with the semigroup of partial permutations of the support lattice of the Coxeter complex for \( S_n \) generated by the partial identities of the supports and the action of \( S_n \).
Let $M$ be a reductive algebraic monoid (e.g. $M_n(K)$).
Let $G$ be the unit group of $M$; it is reductive (e.g. $GL_n(K)$).
Let $T$ be a maximal torus and $B$ a Borel subgroup.
Let $W = N(T)/T$ be the Weyl group.
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**Theorem (Renner)**

$M = \bigsqcup_{r \in R} BrB$ where $R = \overline{N(T)}/T$ is a finite inverse monoid with unit group $W$.

$R$ is called the Renner monoid of $M$. For $M_n(K)$, the Renner monoid is the rook monoid $R_n$. 
1. Find an alternative basis for an inverse semigroup algebra.
2. Use this basis to identify the algebra as a direct product of matrix algebras over group algebras.
3. Compute explicitly the irreducible representations.
4. Give a combinatorial method to compute multiplicities of irreducible constituents in an arbitrary representation.
5. Discuss applications to random walks.
The idempotents of $I_n$ are the partial identities $1_X$ with $X \subseteq [n]$.

They form a lattice isomorphic to $2^X$ as $1_X \cdot 1_Y = 1_{X \cap Y} = 1_Y \cdot 1_X$. 
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$I_n$ can be ordered by $\sigma \leq \tau$ if $\sigma$ is a restriction of $\tau$.

The set of elements $\sigma \in I_n$ with domain and range $X$ is a group isomorphic to $S_X$.

This structure is common to all inverse semigroups.
Let $S$ be an inverse semigroup and $E(S)$ its set of idempotents.

- $E(S)$ is a commutative semigroup.
- $E(S)$ is a meet-semilattice ordered by $e \leq f$ if $e = ef$.
- The ordering on $E(S)$ extends to $S$ by $s \leq t$ if $s \in tE(S)$.
- If $e \in E(S)$, then $G_e = \{s \in S \mid ss^{-1} = e = s^{-1}s\}$ is a group called the maximal subgroup at $e$.
- It is the unit group of $eSe$. 
Idempotents $e, f \in E(S)$ are isomorphic if $\exists s \in S$ such that $s^{-1}s = e$ and $ss^{-1} = f$.

We represent this by an arrow $s^{-1}s \rightarrow ss^{-1}$ and write $\text{dom}(s) = s^{-1}s$ and $\text{ran}(s) = ss^{-1}$.

If $e \cong f$, then $G_e \cong G_f$. 
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One can in fact form a groupoid with objects $E(S)$ and arrows $s^{-1}s \rightarrow ss^{-1}$.

Composition is given by

$$e \xrightarrow{s} f \xrightarrow{t} e' = e \xrightarrow{st} e'.$$
Let $K$ be a field and $S$ a finite inverse semigroup. Let $\mu$ be the Möbius function of the poset $(S, \leq)$. Define, for $s \in S$, an element of $KS$ by

\[ s = \sum_{t \leq s} t \mu(t, s). \]

By Möbius inversion, $s = \sum_{t \leq s} t$. So the $s$ form a basis for $S$.

Theorem (BS)

The basis \{s | s \in S\} satisfies
\[ s \cdot t = \begin{cases} st & \text{dom}(s) = \text{ran}(t) \\ 0 & \text{otherwise}. \end{cases} \]
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$$\bar{s} = \sum_{t \leq s} t \mu(t, s).$$

By Möbius inversion, $s = \sum_{t \leq s} \bar{t}$. So the $\bar{s}$ form a basis for $S$. 

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The Groupoid Basis

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**Theorem (BS)**

The basis $\{s \mid s \in S\}$ satisfies

$$s \cdot t = \begin{cases} 
st & \text{dom}(s) = \text{ran}(t) \\
0 & \text{otherwise}.
\end{cases}$$
From the theorem, it follows \( \{ \overline{e} \mid e \in E(S) \} \) is a set of orthogonal idempotents summing to 1.

Moreover, \( \overline{e}KSe \cong \overline{f}KSe \) if and only if \( e \cong f \).

\( \overline{e}KSe \cong KG_e \).

Let \( e_1, \ldots, e_r \) be a transversal to the set of isomorphism classes of idempotents of \( S \).

Let \( n_i \) be the number of idempotents isomorphic to \( e_i \).
Orthogonal Idempotents and a Decomposition

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- Moreover, \( \overline{e}KS \cong \overline{f}KS \) if and only if \( e \cong f \).
- \( \overline{e}KS\overline{e} \cong KG_e \).
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KS \cong \prod_{i=1}^{r} M_{n_i}(KG_{e_i}).
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**Proof.**

\[
KS \cong \prod_{i=1}^{r} \text{End}(n_i\overline{e}_iKS) \cong \prod_{i=1}^{r} M_{n_i}(\overline{e}_iKS\overline{e}_i) \cong \prod_{i=1}^{r} M_{n_i}(KG_{e_i}).
\]
The Algebra of $I_n$

For $I_n$, we can take as a transversal $\{1_{[i]} \mid i = 0, \ldots, n\}$.

Then $G_{1_{[i]}} \cong S_i$ and $n_i = \binom{n}{i}$.

So $KI_n \cong \prod_{i=0}^{n} M_{\binom{n}{i}}(KS_i)$.

The corresponding central idempotents are

$$e_i = \sum_{|X|=i} \sum_{Y \subseteq X} (-1)^{|X|-|Y|} 1_Y$$

This explicit decomposition for $KI_n$ was first discovered by Solomon.

In general, the idempotents of a Renner monoid form the face lattice of a rational polytope. Hence the Möbius function is particularly nice in this context.
Munn and Ponizovskii showed in the fifties that the algebra of an inverse semigroup has an ideal series whose successive quotients are the $M_{n_i}(K G_{e_i})$. This implies our decomposition.

But it is not good enough to compute multiplicities of irreducible constituents.

Solomon obtained the explicit decomposition, but did not use it to compute multiplicities.

After I introduced the groupoid basis, it was exploited by Rockmore and Malandro to develop Fast Fourier Transforms for the symmetric inverse monoid.
We retain our previous notation.

$\mathbb{C} S$ is Morita equivalent to $\mathbb{C} G_1 \times \cdots \times \mathbb{C} G_r$.

So $\text{Irr}(S) \cong \bigsqcup_{i=1}^r \text{Irr}(G_i)$.

Let $\theta$ be a character of $S$ and let $\chi$ be an irreducible character of $G_i$.

The associated irreducible character of $S$ is denoted $\chi^*$.

For $f \leq E(S)$, define $\theta_f(s) = \theta(sf)$. 

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**Theorem (BS)**

The multiplicity of \( \chi^* \) in \( \theta \) is given by

\[
\sum_{f \leq e} \langle \chi, \theta_f \rangle \mu(f, e).
\]
Tensor Powers

- Let $G$ be a finite group.
- $G \wr S_n$ acts naturally on $|G| \times [n]$.
- Let $\theta$ be the character of the associated representation and let $\theta^p$ be its $p^{th}$-tensor power.
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**Theorem (BS)**

Let $\chi \in \text{Irr}(G \wr S_r)$. Then the multiplicity of $\chi^*$ in $\theta^p$ is

$$\frac{1}{|G|^{r-p}\deg(\chi)}S(p, r)$$

where $S(p, r)$ is the Stirling number of the second kind.

- This generalizes a result of Solomon for $|G| = 1$, but even in this case our proof is easier as Solomon used a more complicated method to compute multiplicities.
Triangularizable Semigroups

Theorem (AMSV)

The following are equivalent for a finite semigroup $S$:

1. $CS$ is basic;
2. All irreducible representations of $S$ have degree 1;
3. $S$ admits a faithful representation by upper triangular matrices;
4. There is a morphism $\varphi : S \rightarrow T$ with $T$ a commutative inverse semigroup such that the induced map $\bar{\varphi} : CS \rightarrow CT$ is the semisimple quotient;
5. All subgroups of $S$ are abelian and there exists $n > 0$ such that regular elements satisfy $x^n = x$ and products of idempotents satisfy $x^n = x^{n+1}$.

Semigroups satisfying these conditions are called triangularizable.
Let $\pi = \sum_{m \in M} \pi_m m$ be a probability measure on a finite monoid $M$.

Fix a minimal right ideal $R$ of $M$.

For $r_1, r_2 \in R$, let $T_{r_1r_2}$ be the probability $r_1 m = r_2$ if $m \in M$ is chosen according to $\pi$.

Let $T = (T_{r_1r_2})$ be the transition matrix. Then $T_{r_1r_2}^n$ is the probability of going from $r_1$ to $r_2$ on the $n^{th}$-step of the walk.
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What is the spectrum of $T$?
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What is the spectrum of \( T \)?

Can you compute the stationary distribution (the probability vector with eigenvalue 1)?
Random Walks on Monoids

- Let $\pi = \sum_{m \in M} \pi m m$ be a probability measure on a finite monoid $M$.
- Fix a minimal right ideal $R$ of $M$.
- For $r_1, r_2 \in R$, let $T_{r_1 r_2}$ be the probability $r_1 m = r_2$ if $m \in M$ is chosen according to $\pi$.
- Let $T = (T_{r_1 r_2})$ be the transition matrix. Then $T_{r_1 r_2}^n$ is the probability of going from $r_1$ to $r_2$ on the $n^{th}$-step of the walk.
- What is the spectrum of $T$?
- Can you compute the stationary distribution (the probability vector with eigenvalue 1)?
- Diaconis did this for abelian groups, Brown did this for idempotent semigroups (bands), following earlier work of Bidigare, Hanlon and Rockmore for face semigroups of hyperplane arrangements.
Diaconis-Brown trick

- $CR$ is a right ideal in $CM$. Let $\rho : M \to M_{|R|}(\mathbb{C})$ be the associated matrix representation with respect to the basis $R$.
- Key observation: $T = \rho(\pi)$.
- Suppose now $M$ is triangularizable. Taking a basis adapted to a composition series for $CR$ yields
Diaconis-Brown trick

- $\mathbb{C}R$ is a right ideal in $\mathbb{C}M$. Let $\rho : M \to M_{|R|}(\mathbb{C})$ be the associated matrix representation with respect to the basis $R$.
- Key observation: $T = \rho(\pi)$.
- Suppose now $M$ is triangularizable. Taking a basis adapted to a composition series for $\mathbb{C}R$ yields

$$
\rho \sim \begin{pmatrix}
\chi_1 & 0 & \cdots & 0 \\
* & \chi_2 & 0 & \\
& * & \ddots & 0 \\
& & * & \ddots & 0 \\
& & & * & \cdots & \chi_{|R|}
\end{pmatrix}
$$

- $T$ has an eigenvalue $\lambda_\chi = \sum_{m \in M} \pi_m \chi(m)$ associated to each $\chi \in \text{Irr } M$.
- The multiplicity of $\lambda_\chi$ is the multiplicity of $\chi$ in $\rho$.
- Our work on inverse semigroups allows us to explicitly determine these.
Choose an idempotent transversal $\mathcal{E} = \{e_1, \ldots, e_r\}$ to the set of isomorphism classes of $E(M)$.

$Irr\ M = \bigsqcup Irr\ G_{e_i}$ ($G_{e_i} =$ unit group of $e_iMe_i$).

If $\chi \in Irr\ G_{e_i}$, then
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\( (G_{e_i} = \text{unit group of } e_i Me_i) \).

If \( \chi \in \text{Irr } G_{e_i} \), then

\[ \chi(e_i) = \sum_{e_i \in MmM} \pi_m \chi(e_i me_i). \]

The set \( \mathcal{E} \) is a meet-semilattice with respect to the ordering \( e_j \leq e_i \) if \( Me_j M \subseteq Me_i M \). Let \( \mu \) be its Möbius function.

The multiplicity of \( \lambda_\chi \) is given by
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The multiplicity of $\lambda_\chi$ is given by

$$\sum_{e_j \leq \mathcal{E} e_i} \langle \chi, \varphi_j \rangle \mu(e_j, e_i)$$

where $\varphi_j(g)$ is the number of fixed points of $e_j ge_j$ on $R$.

This recovers the results of Diaconis on abelian groups and of Brown on bands.