Étale groupoids and inverse semigroups

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Ruy Exel suggested the following paradigm for the study of “combinatorial $C^\ast$-algebras.”
- Start with a combinatorial object (e.g., a graph or a tiling).
- Construct an inverse semigroup.
- From the semigroup construct an étale groupoid.
- From the groupoid construct a $C^\ast$-algebra.

One can hope to use the algebra to understand the combinatorial object (as in the case of tilings);
Or use the combinatorial object to understand the algebra (as in the case of graphs).
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For instance, Morita equivalent groupoids have strongly Morita equivalent algebras.

The (partial action) semidirect product of a group with a space yields a (partial action) cross product of the group with a commutative $C^*$-algebra.

The algebra of a groupoid of germs of an inverse semigroup action on a space translates to an inverse semigroup cross product algebra.
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It seems to be less well known what happens when passing from an inverse semigroup to a groupoid.

Does Morita equivalence pass through to the groupoids?

Do (partial action) semidirect products become (partial action) semidirect products?

Functoriality seems not to have been addressed.

It is also not universally agreed upon which groupoid to assign to an inverse semigroup.

There are Paterson’s universal groupoid, Kellendonk’s groupoid and Exel’s tight groupoid.

Here we will focus on Paterson’s groupoid, as the others are reductions. All of our results so far are valid for any of these groupoids.
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Inverse semigroups are the abstract counterparts of pseudogroups of partial homeomorphisms of a space.

If $X$ is a topological space, then $I_X$ is the pseudogroup of all homeomorphisms between open subsets of $X$.

Composition is defined where it makes sense: if $f : U \to V$ and $g : U' \to V'$, then

$$f \circ g : g^{-1}(U \cap V') \to f(U \cap V').$$

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Formally, an inverse semigroup is a semigroup $S$ such that for all $s \in S$, there is a unique element $s^*$ such that

$$ss^*s = s, \quad s^*ss^* = s^*$$

Munn and Penrose proved that the idempotents of $S$ commute and hence $E(S)$ is a subsemigroup.

$E(S)$ is ordered by $e \leq f$ iff $ef = e$ and then $ef = e \land f$.

So $E(S)$ is a meet semilattice.

The order extends to $S$ by putting $s \leq t$ if $s = te$ for some idempotent $e$.

For $I_X$, the ordering is restriction.
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More on inverse semigroups

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For $I_X$, the ordering is restriction.
The maximal group image

- Identifying two elements of $S$ with a common lower bound (i.e., the same germ) yields a congruence.
- The quotient is a group $G(S)$ called the maximal group image.
- The canonical projection $\sigma: S \rightarrow G(S)$ is universal to groups.
- Clearly $E(S) \subseteq \sigma^{-1}(1)$.
- $S$ is called $E$-unitary if $E(S) = \sigma^{-1}(1)$.
- $S$ is called $F$-inverse if $eSf \cap \sigma^{-1}(g)$ has a maximum element for all $e, f \in E(S)$ and $g \in G(S)$.
- $F$-inverse semigroups are $E$-unitary.
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Extreme examples

- If $E$ is a meet semilattice, then $E$ is an inverse semigroup of "projections."
- Meet semilattices are to inverse semigroups as spaces are to étale groupoids.
- A group is an inverse semigroup.
- If a group $G$ acts on a semilattice $E$, the semidirect product $E \rtimes G$ is an $F$-inverse semigroup.
- If $G$ acts partially on $E$, then $E \rtimes G$ is $E$-unitary and by a theorem of McAlister all $E$-unitary inverse semigroups arise in this way.
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More interesting examples

- The **bicyclic** monoid is the inverse semigroup $B$ generated by a unilateral shift and its adjoint.
- Its semilattice of idempotents is $(\mathbb{N}, \geq)$.
- It is $E$-unitary with maximal group image is $\mathbb{Z}$ (given by the Fredholm index).
- The **polycyclic** (or Cuntz) inverse semigroup is given by the presentation $P_X = \langle X \mid x^*y = \delta_{x,y} \rangle$ (as an inverse semigroup with 0).
- It is a (congruence) simple inverse semigroup and can be identified with the inverse semigroup generated by the generators of the Cuntz algebra.
- Similarly, there is a graph inverse semigroup associated to any directed graph which is analogous to the graph $C^*$-algebra.
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$M_A(G)$ consists of all equivalence classes of triples $[g, \Lambda, h]$ where $\Lambda$ is a finite connected subgraph of $\Gamma$ and $g, h$ are vertices of $\Lambda$.

Equivalence is up to translation by $G$.

The product is given by

$$[g, \Lambda, h][g', \Lambda', h'] = [g, \Lambda \cup h\Lambda', hh'].$$ 

$M_A(G)$ is $E$-unitary with maximal group image $G$.

If $G$ is free on $A$, then $M_A(G)$ is a free inverse monoid on $A$.

Free inverse monoids classify graph immersions over a bouquet.
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Free inverse monoids classify graph immersions over a bouquet.
The Meakin-Margolis-Munn construction

- Let $G$ be a group with generators $X$ and corresponding Cayley graph $\Gamma$.
- $M_A(G)$ consists of all equivalence classes of triples $[g, \Lambda, h]$ where $\Lambda$ is a finite connected subgraph of $\Gamma$ and $g, h$ are vertices of $\Lambda$.
- Equivalence is up to translation by $G$.
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Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$.

Kellendonk’s tiling semigroup has elements 0 and equivalence classes $[t, P, u]$ where $P$ is a pattern and $t, u$ are tiles of $P$.

Equivalence is up to translation in $\mathbb{R}^d$.

The product $[t, P, u][t', P', u']$ is non-zero iff there are $v, v' \in \mathbb{R}^d$ such that $u + v = t' + v'$ and $(P + v) \cup (P' + v')$ is a pattern.

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Then $S_X$ is an $F$-inverse monoid and the maximal group image is the group of birational automorphisms of $X$.

When $X = \mathbb{P}^1$, then $S_X$ is the Möbius inverse semigroup and $G(S_X)$ is the group of linear fractional transformations.

Let $\{a, b\}^*$ be a free monoid on $\{a, b\}$ and $S$ the inverse semigroup of isomorphisms between finitely generated essential right ideals of $\{a, b\}^*$.

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The same semigroup was discovered earlier by Birget and Rhodes in a different context.

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An inverse semigroups with 0 cannot be $E$-unitary without being a semilattice.

So one needs an appropriate variation of the concept for this context.

If $S$ is an inverse semigroup with 0, the universal group $U(S)$ is the group with generators $S \setminus \{0\}$ and relations given $s \cdot t = st$ if $st \neq 0$.

The natural map $\sigma: S \setminus \{0\} \to U(S)$ satisfies $\sigma(s)\sigma(t) = \sigma(st)$ whenever $st \neq 0$ and is universal for this property.

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More generally, graph inverse semigroups are strongly 0-$E$-unitary.

Tiling inverse semigroups are strongly 0-$E$-unitary (Lawson/Kellendonk).

It seems most of the inverse semigroups appearing in the $C^*$-algebra literature are strongly 0-$E$-unitary.

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This allows a partial action semidirect product type description of $S$ and turns out to be useful for understanding the $C^*$-algebra of $S$. 

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The groupoid of germs

- An action $S \curvearrowright X$ of an inverse semigroup on a space $X$ is a homomorphism $\theta : S \to I_X$.
- We always assume that $X = \bigcup_{e \in E(S)} X_e$ where $X_e$ is the domain of $e$.
- Note $\theta(s) : X_{s^*s} \to X_{ss^*}$.
- Let us call the action special if $X_e$ is clopen for each $e \in E(S)$.
- The groupoid of germs $S \ltimes X$ has arrow space $(S \times X)/\sim$ where $(s, x) \sim (t, y)$ iff $x = y$ and $\exists u \leq s, t$ with $x \in X_{u^*u}$.
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- We always assume that $X = \bigcup_{e \in E(S)} X_e$ where $X_e$ is the domain of $e$.
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Let $\hat{E}$ be the character space of $E$, that is, the space of non-zero homomorphisms $E \to \{0, 1\}$ equipped with the topology of pointwise convergence.

When working in the category of inverse semigroups with 0, we assume characters preserve 0.

Elements of $\hat{E}$ can also be identified with filters on $E$.

Then $S \curvearrowright \hat{E}$ by putting $\hat{E}_e = \{ \varphi \mid \varphi(e) = 1 \}$ and $s\varphi(f) = \varphi(s^*fs)$ for $\varphi \in \hat{E}_{s^*s}$.

This is a special action.

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Étaté groupoids and inverse semigroups
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A result of Khoshkam and Skandalis

• Khoshkam and Skandalis proved the following result.

**Theorem**

Let $S$ be an $E$-unitary inverse semigroup such that $eSf \cap \sigma^{-1}(g)$ is finitely generated as a downset for all $e, f \in E(S)$ and $g \in G(S)$. Then there is an action $G(S) \curvearrowright X$ such that $G(S)$ is Morita equivalent to $G(S) \ltimes X$.

• $F$-inverse semigroups satisfy the Khoshkam-Skandalis condition since $eSf \cap \sigma^{-1}(g)$ has a maximum.

• The proof uses a generalization of the enveloping action of a partial group action.

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There is another way to view the Khoshkam-Skandalis result.

**Theorem (Milan, BS)**

Let $S$ be an $E$-unitary inverse semigroup. Then there is a partial action of $G(S)$ on $\hat{E}(S)$ such that $\mathcal{G}(S) \cong G(S) \ltimes \hat{E}(S)$.

- Here the partial action semidirect product is as in the sense of Abadie.
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The strongly $0$-$E$-unitary case

- Let $S$ be an inverse semigroup and $I$ an ideal.
- Then $\hat{E}(S/I)$ can be identified with those characters of $E(S)$ vanishing on $I$.
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**Theorem (Milan, BS)**

*If $S$ is strongly $0$-$E$-unitary with universal group $U(S)$, then $\mathcal{G}(S) \cong U(S) \ltimes \hat{E}(S)$ for an appropriate partial action of $U(S)$.***

- A similar result holds for Exel's tight groupoid and explains a number of partial action cross product results in the literature.
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What are conditions on a homomorphism $S \xrightarrow{\varphi} T$ that guarantee that $C^*(S')$ is strongly Morita equivalent to a cross product $C_0(X) \rtimes T$?

In other words, when is $\mathcal{G}(S')$ Morita equivalent to a groupoid of germs of an action of $T$?

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Functoriality

- If $E \xrightarrow{\varphi} F$ is a semilattice homomorphism, then there is a natural cts map $\widehat{F} \rightarrow \widehat{E}$ given by “pulling back.”
- But we want a map the other way — and there is one.
- If $\mathcal{F}$ is a filter on $E$, then $\varphi(\mathcal{F})$ is a filter base on $F$ and hence yields a filter $\varphi_*(\mathcal{F})$.
- However, $\varphi_*$ need not be cts.
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However, $\varphi_*$ need not be cts.
The Alexandrov topology on a poset takes the downsets as the open sets. (It is $T_0$.)

The quasi-compact open sets are the finitely generated downsets.

A map of spaces is coherent if the inverse image of a quasi-compact open is quasi-compact open.

It is a folklore fact from the theory of locales that $\varphi_*$ is cts iff $\varphi$ is locally coherent.

Theorem (Milan, BS)

The universal groupoid is functorial on the category of inverse semigroups with maps that are locally coherent on the semilattices of idempotents.

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Let us say $S \xrightarrow{\varphi} T$ is locally idempotent-pure if $\varphi|_{eSe}$ is idempotent-pure for each $e \in E(S)$.

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First they gave a general condition on a functor $\mathcal{G} \to G$ from a groupoid to a group (a cocycle) that guarantees $\mathcal{G}$ is Morita equivalent to $G \ltimes X$ for an appropriate action $G \rhd X$.

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- $p(x)x = x$;
- $g(hx) = (gh)x$;
- $p(gx) = r(g)$.

The semidirect product $\mathcal{G} \rtimes X$ has arrow space $\mathcal{G} \times_{\mathcal{G}^0} X$.

The unit space is $X$.

One has $x \xrightarrow{(g,x)} gx$.

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Benjamin Steinberg

**Étale groupoids and inverse semigroups**
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Theorem (Milan, BS)

Let \( \varphi: G \to \mathcal{H} \) be a cts functor with \( G \) locally compact and \( \mathcal{H} \) étale. Suppose that:

- the map \( G \to G^0 \times \mathcal{H} \times \mathcal{H}^0 G^0 \) given by
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Benjamin Steinberg
Étale groupoids and inverse semigroups
A Morita equivalence theorem

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Then there is an action \( \mathcal{H} \acts X \) such that \( G \) is Morita equivalent to \( \mathcal{H} \ltimes X \).
To apply the previous situation, we need to identify groupoids of germs as semidirect products.

**Theorem (Milan, BS)**

1. There is a bijection between special actions $T \curvearrowright X$ and actions $\mathcal{G}(T) \curvearrowright X$.
2. Moreover, the groupoid of germs $T \ltimes X$ is isomorphic to the semidirect product $\mathcal{G}(T) \ltimes X$.

For instance, if $\mathcal{G}(T)$ acts on $X$ with $X \xrightarrow{p} \widehat{E}(T)$, then we put $X_e = p^{-1}(\widehat{E}_e)$.

For $x \in X_{t\ast t}$, we define $tx = [t, p(x)]x$.

Conversely, if $T \curvearrowright X$, then $X \xrightarrow{p} \widehat{E}(T)$ is given by the "neighborhood filter:" $p(x) = \{ e \mid x \in X_e \}$.

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Étale groupoids and inverse semigroups
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To prove our main result, one verifies that if $S \xrightarrow{\varphi} T$ is a locally idempotent-pure morphism satisfying the KS condition, then the induced map $\mathcal{G}(S) \rightarrow \mathcal{G}(T)$ satisfies the conditions of our Morita theorem.

Thus $\mathcal{G}(S)$ is Morita equivalent to $\mathcal{G}(T) \rtimes X \cong T \rtimes X$.

Therefore, $C^*(S)$ is strongly Morita equivalent to $C_0(X) \rtimes T$. 
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Let $S$ be an inverse semigroup.

Let $S\text{-Set}$ be the category of sets $X$ equipped with an action $S \times X \rightarrow X$ by total functions such that $SX = X$.

Talwar defined two inverse semigroups $S$ and $T$ to be Morita equivalent if $S\text{-Set}$ is equivalent to $T\text{-Set}$.

For example, if $S$ is an inverse monoid, then $S$ and $T$ are Morita equivalent iff there is an idempotent $e$ with $T = TeT$ and $S \cong eTe$.

An inverse semigroup is $F$-inverse iff it is Morita equivalent to a semidirect product of a group and a semilattice.

Lawson calls $S$ an enlargement of a subsemigroup $T$ if $S = STS$ and $T = TST$.

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I defined inverse semigroups $S$ and $T$ to be strongly Morita equivalent if there is an equivalence bimodule for $S$ and $T$.

By definition, this consists of a set $X$, which is an $(S,T)$-biset equipped with surjective “inner products”

$$\langle -,- \rangle : X \times X \rightarrow S,$$

$$[ -,- ] : X \times X \rightarrow T$$

such that the following axioms hold, where $x, y, z \in X$, $s \in S$, and $t \in T$:

- $\langle sx, y \rangle = s \langle x, y \rangle$
- $[ x, yt ] = [ x, y ] t$
- $\langle y, x \rangle = \langle x, y \rangle^*$
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- $\langle x, x \rangle x = x$
- $x [ x, x ] = x$
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Strong Morita equivalence and groupoids

Theorem (BS)

If $S$ and $T$ are strongly Morita equivalent, then $\mathcal{G}(S)$ and $\mathcal{G}(T)$ are Morita equivalent.

- Every $F$-inverse semigroup $S$ has an enlargement $E \rtimes G(S)$.
- Thus $C^*(S)$ is strongly Morita equivalent to $C_0(\hat{E}) \rtimes G(S)$.
- This yields an alternative proof to that of Khoshkam and Skandalis.
- It is easy to see that strong Morita equivalence implies Morita equivalence.
- I had conjectured the converse is false.

Theorem (Funk, Lawson, BS)

Morita equivalence and strong Morita equivalence are the same.

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Thanks for your attention!
Je vous remercie de votre attention!
Obrigado pela sua atenção!