Model Theory and Rational Dynamics

Alice Medvedev

University of Illinois at Chicago

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This is a story about

- **Dynamical systems** $F : X \to X$.
- Difference equations.
  - $K \geq \mathbb{C}$ is a huge field and $f_i \in K[x]$
  - $\sigma : K \to K$ is a generic field automorphism such that $\sigma|_{\mathbb{C}} = \mathrm{id}|_{\mathbb{C}}$.
- **The definable structure on the solution sets of $\sigma(x) = f_i(x)$, and definable relations between them, are intimately related to the dynamical properties of $F(x_1, \ldots, x_n) = (f_1(x_1), \ldots f_n(x_n))$.**

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- **Algebraic** dynamical systems $F : X \to X$, where X is a variety and F is a morphism of varieties.
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- **Coordinate-wise polynomial** dynamical systems on $\mathbb{C}^n$
  $F : \mathbb{C}^n \to \mathbb{C}^n$ where $F(x_1, \ldots, x_n) = (f_1(x_1), \ldots f_n(x_n))$
  for some polynomials $f_i \in \mathbb{C}[x]$.
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Algebraic dynamical properties

- What subsets $V \subset \mathbb{C}^n$ are F-**invariant**, i.e. $F(V) \subset V$?
- What do the **orbits** of F look like?

$$\mathcal{O}_F(a) := \{a, F(a), F(F(a)), \ldots F^{\circ n}(a), \ldots\}$$

- These two questions are not unrelated.
  - If $a \in V$ and V is F-invariant, $\mathcal{O}_F(a) \subset V$.
  - The Zariski-closure V of an F-orbit is F-invariant.

$$V := \text{Zeros}(\{Q \mid \forall m\, Q(F^{\circ m}(a)) = 0\})$$

  - If a does not lie on any F-invariant subvariety, then $\mathcal{O}_F(a)$ is Zariski-dense in $\mathbb{C}^n$.

Algebraic dynamical properties

- What **subvarieties** $V \subset \mathbb{C}^n$ are F-invariant: $F(V) \subset V$? (subvariety = solution set of some polynomial equations)
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Algebraic dynamical properties

- What subvarieties $V \subset \mathbb{C}^n$ are F-invariant: $F(V) \subset V$? (subvariety = solution set of some polynomial equations)
- What **algebraic** relations hold on an F-orbit of $a \in \mathbb{C}^n$?

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- For what $Q \in \mathbb{C}[x_1, \ldots x_n]$ does $Q(F^{\circ m}(a)) = 0$ for all m?

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Zhang's Conjecture

Question:

Conjecture (Zhang)

Let K be a number field and $f : X \to X$ a polarizable dynamical system over K. Then some point in $X(K^{alg})$ has a Zariski dense orbit.

If a does not lie on any F-invariant subvariety, then $\mathcal{O}_F(a)$ is Zariski-dense in $\mathbb{C}^n$.

Answer:

Theorem (M.-Scanlon)

Coordinatewise polynomial dynamical systems on $\mathbb{C}^n$ have Zariski-dense orbits because they have very few invariant subvarieties, unless...

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Examples: $(f, f, \dots f)$-invariant subvarieties.

- (unary) If $f(a) = a$, then $\text{Zeros}(x_i - a)$.
- (binary) $\text{Zeros}(x_2 - x_1)$ (diagonal)
- (binary) $\text{Zeros}(x_2 - f(x_1))$ (graph of f)
- (ternary) $\text{Zeros}(x_3 - f(f(x_1)),\ f^{\circ 4}(x_2) - f^{\circ 7}(x_3))$ but
- (really ternary) If $f(x) = x^3$, $\text{Zeros}(x_1 x_2 - x_3)$.
- ((n-1)ary) If $f(x) = x + 2$, for $Q \in \mathbb{C}[y_1, \dots y_{n-1}]$ $\text{Zeros}(Q(x_2 - x_1, x_3 - x_1, \dots x_n - x_1))$.

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- (really ternary) If $f(x) = x^3$, $\text{Zeros}(x_1 x_2 - x_3)$. **Graph of group operation invariant under homomorphism.**
- ((n-1)ary) If $f(x) = x + 2$, for any $Q \in \mathbb{C}[y_1, \ldots y_{n-1}]$ $\text{Zeros}(Q(x_2 - x_1, x_3 - x_1, \ldots x_n - x_1))$: $\mathbf{F \text{ fixes } x_i - x_j}$.

Essential arity: fiber products

- A system S of polynomial equations in n variables is **essentially k-ary** if there are $A_i \subsetneq \{1, 2, \ldots n\}$ with $|A_i| \leq k$ and polynomial systems S_i in variables from A_i such that $\text{Zeros}(S) = \text{Zeros}(\cup_i S_i)$. Then the set $\text{Zeros}(S)$ is a **fiber product** of the sets $\text{Zeros}(S_i)$.
- Ternary, essentially binary: (f, f, f)-invariant $\text{Zeros}(x_3 - f(f(x_1)),\ f^{\circ 4}(x_2) - f^{\circ 7}(x_3))$.
- Ternary, essentially ternary (group): $\text{Zeros}(x_1 x_2 - x_3)$ is (x^5, x^5, x^5)-invariant.
- Mysterious essentially ternary: $\text{Zeros}(x_3 x_2 x_1 - x_3 x_2 - 1)$ is $(x^3 - 3x, x^3, x^3)$-invariant.
- If $f_i(x) = x$ for k distinct i, there are many essentially k-ary F-invariant subvarieties .

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Linear change of variables.

- If $f_2 = \ell \circ f_1 \circ \ell^{-1}$ for a linear ℓ
  - $\text{Zeros}(x_2 - \ell(x_1))$ is (f_1, f_2)-invariant.
  - If V is (f_1, f_1)-invariant, $\{(\ell(a), b) \mid (a, b) \in V\}$ is (f_1, f_2)-invariant.
- f **linearly related to** g: $g = \ell \circ f \circ \ell^{-1}$ for some linear ℓ.
- If all f_i are linearly related to the same f, F-invariant sets come from $(f, f, \ldots, f)$-invariant ones.
- If $G = L \circ F \circ L^{-1}$ for linear $L : \mathbb{C}^n \to \mathbb{C}^n$, V is F-invariant iff $L(V)$ is G-invariant.
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A trichotomy, toward classification.

Summarizing experimental observations:

- Any F admits unary and binary invariant subvarieties.
- If at least three f_i are linearly related to monomials of Chebyshevs all of the same degree, there are essentially ternary F-invariant subvarieties.
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Theorem (M.-Scanlon)

All F-invariant subvarieties are essentially unary or binary, unless

- *at least three f_i are linearly related to monomials of Chebyshev polynomials all of the same degree, and then there are essentially ternary F-invariant subvarieties; or*
- *F is linearly related to G with at least k of the $g_i(x) = x$, and then there are essentially k-ary ones.*

We then classify n-ary essentially n-ary invariant subvarieties. This suffices for many purposes including Zhang's conjecture.

A generic field automorphism $\sigma : K \to K$.

(K, σ) is a *difference-closed field*: systems of difference polynomial equations that have solutions in some extension of (K, σ) have a solution in (K, σ).

- An F-invariant subvariety V is a definable relation amongst the sets $f_i^\sharp := \{a \mid \sigma(a) = f_i(a)\}$.
- Nonorthogonality: $f_i^\sharp \not\perp f_j^\sharp$ if there is an essentially binary definable relation between $f_i^\sharp$ and $f_j^\sharp$.
- If there is an essentially n-ary $(f_1, f_2, \ldots, f_n)$-invariant subvariety of $\mathbb{C}^n$, then $f_i^\sharp \not\perp f_j^\sharp$ for all $i, j \leq n$.
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The Zilber Trichotomy

Our main tool:

Theorem (M.-Scanlon)

In a difference-closed field, polynomials come in three flavors:

- **Fieldlike** *linear f are all nonorthogonal to each other.*
- **Grouplike** *f, linearly equivalent to monomials or Chebyshev polynomials , are nonorthogonal iff they have the same degree, and all definable relations among such $f^\sharp$ are essentially ternary.*
- *All other polynomials are* **trivial***: all definable relations among such $f^\sharp$ are essentially unary or binary.*

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Theorem (M.-Scanlon)

In a difference-closed field, **definable sets $\mathbf{f}^\sharp$ for polynomial $\mathbf{f} \in \mathbb{C}[\mathbf{x}]$** *come in three flavors:*

- *Fieldlike linear f are all nonorthogonal to each other.*
- *Grouplike f,* **linearly equivalent to monomials or Chebyshev polynomials**, *are nonorthogonal iff they have the same degree, and all definable relations among such $f^\sharp$ are essentially ternary.*
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We stand on the shoulders of giants: only **these** are new results.

The Zilber Trichotomy

Giants' shoulders:

Theorem (Chatzidakis, Hrushovski, and Peterzil)

(Zilber Trichotomy for difference-closed fields.)
Any Lascar rank 1 *type in ACFA is exactly one of the following:*

- *(fieldlike) nonorthogonal to a generic type of a fixed field of a definable automorphism;*
- *(grouplike) nonorthogonal to a generic type of a rank* 1 *definable modular subgroup of an algebraic group;*
- *(trivial) model-theoretic algebraic closure is essentially unary.*

Back to algebraic dynamics

Theorem (M.-Scanlon)

Any F-invariant subvariety of $\mathbb{C}^n$ is a fiber product of

- (f_i, f_j)*-invariant place curves;*
- (f_i, f_j, f_k)*-invariant surfaces coming from the graph of multiplication, when all three are linearly equivalent to monomials of Chebyshev polynomials of the same degree;*
- *subvarieties that only involve those x_i for which f_i are linear.*

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Linear dynamics are completely understood.
We then completely characterize $(f, \hat{f})$-invariant place curves as compositions of graphs of g such that $h_i \circ g = g \circ h_{i+1}$ or vice versa, with $f = h_1$ and $\hat{f} = h_n$, refining Ritt's Theorem.

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Theorem (M.-Scanlon)

There is a Zariski-dense F-orbit unless F is linearly related to G with $g_i(x) = x$ for some i.

Thank you

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