Lacunary hyperbolic groups

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ASYMPTOTIC CONES OF FINITELY GENERATED GROUPS
Ultrafilters

**Definition**

A *(non-principal)* ultrafilter $\omega$ is a function from the set of all subsets of $\mathbb{N}$ to \{0, 1\} such that

1. $\omega(A \cup B) = \omega(A) + \omega(B)$ $\forall A, B \subseteq \mathbb{N}$ such that $A \cap B = \emptyset$.
2. $\omega(\mathbb{N}) = 1$.
3. $\omega(F) = 0$ whenever $F \subseteq \mathbb{N}$ is finite.

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of numbers, $\omega$ an ultrafilter. Then

$$\lim_{\omega} x_n = a \iff \omega(\{n \in \mathbb{N} : |x_n - a| < \epsilon\}) = 1 \quad \forall \epsilon > 0.$$ 

**Theorem**

*For every bounded sequence of real numbers $(x_n)$ and any ultrafilter $\omega$, there exists unique $\lim_{\omega} x_n$.***

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Lacunary hyperbolic groups
Let $X$ be a metric space. Fix an observation point $e = (e_n)$, $e_n \in X$, an increasing scaling sequence of positive integers $d = (d_n)$, and an ultrafilter $\omega$.

**Definition**

Given two sequences $x = (x_n)$ and $y = (y_n)$ of elements of $X$, set

$$\text{dist}(x, y) = \omega \lim_{n \to \infty} \frac{\text{dist}(x_n, y_n)}{d_n}.$$  

Further,

$$x \sim y \iff \text{dist}(x, y) = 0.$$  

The asymptotic cone of $X$ with respect to $e$, $d$, and $\omega$ is

$$\text{Con}^\omega (X, d, e) = \{ x = (x_n) | \text{dist}(x, e) < \infty \} / \sim$$

with the distance induced by $\text{dist}$. If $X$ is a group endowed with a word metric, the asymptotic cone in independent of $e$ and is denoted by $\text{Con}^\omega (X, d)$. 

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Examples

1. If $X$ is a finite group, then $\text{Con}^\omega(X, d)$ is a point $\forall \ d, \omega$. 

2. $\forall \ d, \omega, \text{Con}^\omega(\mathbb{Z}^n, d) = \mathbb{R}^n$ with ‘Manhattan metric’: for $a = (a_i), b = (b_i) \in \mathbb{R}^n$,

\[ \text{dist}(a, b) = \sum_{i=1}^{n} |a_i - b_i|. \]

More generally, for any finitely generated nilpotent group $N$, $\text{Con}^\omega(N, d)$ is homeomorphic to $\mathbb{R}^n \forall \ d, \omega$, where $n$ is the Hirsch number of $N$.

3. If $G$ is a $\delta$ hyperbolic group, then $(G, \frac{1}{r}\text{dist})$ is $\delta/r$-hyperbolic. Hence $\forall \ d, \omega, \text{Con}^\omega(G, d)$ is 0-hyperbolic, i.e., is a real tree.
Varying scaling sequences and ultrafilters

**Theorem (Thomas–Velickovic)**

There exists a group $G$ and two ultrafilters $\omega_1$, $\omega_2$, such that $\text{Con}^{\omega_1}(G, (n))$ is a real tree while $\text{Con}^{\omega_2}(G, (n))$ is not simply connected.

**Theorem (Drutu–Sapir)**

There exists a finitely generated group with uncountably many non-homeomorphic asymptotic cones.

**Theorem (Olshanskii–Sapir)**

There exists a finitely presented group with at least 2 non-homeomorphic asymptotic cones.

**Observation.** $\forall$ metric space $X$, observation point $e$, increasing sequence of integers $d = (d_n)$, and ultrafilter $\omega$, $\exists$ an ultrafilter $\xi$ such that $\text{Con}^\omega(X, (n), e)$ is isometric to $\text{Con}^{\xi}(X, (n), e)$.
Asymptotic cones of hyperbolic groups

Theorem (Gromov)
A finitely generated group $G$ is hyperbolic iff all asymptotic cones of $G$ are $\mathbb{R}$-trees.

Theorem (M. Kapovich–Kleiner)
If $G$ is a finitely presented group and at least one asymptotic cone of $G$ is an $\mathbb{R}$-tree, then $G$ is hyperbolic.

Definition
A finitely generated group $G$ is lacunary hyperbolic if at least one asymptotic cone of $G$ is a real tree.
LACUNARY HYPERBOLIC GROUPS: A CHARACTERIZATION AND EXAMPLES
Equivalent definitions of lacunary hyperbolic groups

Given a homomorphism $\alpha : G \to H$ and a generating set $S$ of $G$, we define the *injectivity radius* $IR_S(\alpha)$ of $\alpha$ with respect to $S$ to be the radius of the largest ball in $G$ on which $\alpha$ is injective.

**Theorem (Olshanskii–Osin–Sapir)**

Let $G$ be a finitely generated group. Then the following conditions are equivalent.

1. $G$ is lacunary hyperbolic.
2. There exists a scaling sequence $d = (d_n)$ such that $\text{Con}^\omega (G, d)$ is an $\mathbb{R}$–tree for any ultrafilter $\omega$.
3. $G$ is the direct limit of a sequence of hyperbolic groups $G_i$ generated by finite sets $S_i$ and epimorphisms

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} \ldots,$$

where $\alpha_i(S_i) = S_{i+1}$, $G_i$ is $\delta_i$–hyperbolic with respect to $S_i$, and $\delta_i = o(IR_{S_i}(\alpha_i))$. 

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Limits of hyperbolic groups that are not lacunary hyperbolic

**Theorem (Drutu–Sapir)**

Let $G$ be a non-elementary finitely generated group. If $\text{Con}^\omega(G, d)$ has a cut point, then $\prod^\omega G$ contains a non-abelian free subgroup.

**Corollary**

Non-elementary groups satisfying a law are not lacunary hyperbolic.

**Example.** The wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ and the free Burnside group $B(m, n)$ are directed limits of hyperbolic groups, but are not lacunary hyperbolic.
We say that a subset \( L \subseteq \mathbb{N} \) is **sparse**, if for any \( \varepsilon > 0 \), there exists a segment \( I = [a, b] \subseteq [1, +\infty) \) such that \( I \cap L = \emptyset \) and \( a/b < \varepsilon \).

**Proposition (Olshanskii–Osin–Sapir)**

Let \( G = \langle X \mid R \rangle \) be a group presentation, where \( X \) is finite and \( R \) satisfies the \( C'(\lambda) \) small cancellation condition for some \( \lambda < 1/6 \). Then \( G \) is lacunary hyperbolic if and only if the set \( \{|R| \mid R \in R\} \) is sparse.

**Corollary**

There are lacunary hyperbolic groups \( H_1, H_2 \) such that \( H_1 * H_2 \) is not lacunary hyperbolic.

*Idea of the proof:* The union of two sparse sets is not necessarily sparse.
Let $H \leq G = \langle S \rangle$, $\#S < \infty$. We say that $H$ has exponential growth in $G$

$$\# \{ h \in H \mid |h|_S \leq n \} \geq d^n$$

for some $d > 1$ and all $n$ big enough.

**Theorem (Olshanskii–Osin–Sapir)**

Let $G$ be a lacunary hyperbolic group. Then:

- Every finitely presented subgroup of $G$ embeds in a hyperbolic group.
- Every undistorted subgroup of $G$ is lacunary hyperbolic.
- No subgroup of $G$ has bounded torsion and exponential growth in $G$.

**Corollary**

Lacunary hyperbolic groups can not contain Baumslag–Solitar subgroups, $B(m, n)$, $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$, etc.
GROUPS WITH TREE-GRADED ASYMPTOTIC CONES
Tree-graded spaces

Definition

A geodesic metric space $X$ is **tree-graded** with respect to a collection of connected subsets $\mathcal{P}$ (called **pieces**) if:

1. Any two distinct pieces intersect by at most one point.
2. Every non-trivial simple geodesic triangle in $X$ is contained in a single piece.

Theorem (Osin–Sapir)

*If a finitely generated group $G$ is hyperbolic relative to a collection of subgroups $\{H_1, \ldots, H_n\}$, then for any scaling sequence $d = (d_n)$ and any ultrafilter $\omega$, $\text{Con}^\omega(G, d)$ is tree-graded with respect to pieces homeomorphic to cones $\text{Con}^\omega(H_i, d)$.***
**Corollary**

If a finitely generated group $G$ is hyperbolic relative to a lacunary hyperbolic subgroup, then $G$ is lacunary hyperbolic itself.

**Problem**

Suppose that a finitely presented group $G$ is hyperbolic relative to a subgroup $H$. Is $H$ finitely presented?

An ultimate negation would be:

*Every recursively presented group $H$ embeds into a finitely presented group $G$ which is hyperbolic relative to $H$.*

The Corollary implies that this is not true.
Constricted groups

Definition (Drutu-Sapir)

A group $G$ is constricted if all asymptotic cones of $G$ have cut points.

Known examples of constricted groups.

1. Groups that are hyperbolic relative to proper subgroups (Osin–Sapir).
2. Mapping class groups (Behrstock).

Asymptotic cones of constricted groups are naturally tree-graded with respect to maximal subsets without cut points. An action of a group on a tree-graded space under some mild assumptions leads to an action on an $\mathbb{R}$-tree. This allows to apply the Rips theory to study constricted groups.
Some questions about constricted groups

The following natural questions were open until now:

1. Does every non–elementary constricted group contain a free non–abelian subgroup?
2. Is every infinite constricted group non-simple?
3. Can a constricted group be periodic?
If a geodesic space $X$ is tree-graded with respect to a collection of circles whose diameters are uniformly bounded from above and from below, we call $X$ a *circle-tree*.

Call a group $G$ *strongly lacunary hyperbolic* if every asymptotic cone of $G$ is either an $\mathbb{R}$–tree or a circle–tree.

\[
\left\{ \text{constricted groups} \right\} \supset \left\{ \text{strongly lacunary hyperbolic groups} \right\} \subset \left\{ \text{lacunary hyperbolic groups} \right\}
\]

**Theorem (Olshanskii–Osin–Sapir)**

1. There exist infinite periodic strongly lacunary hyperbolic groups.
2. There exist strongly lacunary hyperbolic Tarskii Monsters (i.e., non-elementary groups all of whose proper subgroups are cyclic). In particular these groups are simple.
1. We introduce a new small cancellation condition \( Q(\alpha, K) \) for graded presentations, i.e., presentations of the form

\[
\left\langle S \mid \bigcup_{i=0}^{\infty} R_i \right\rangle
\]

Roughly it requires the group \( G_0 = \left\langle S \mid R_0 \right\rangle \) to be hyperbolic and sets \( R_{n+1} \) to satisfy some small cancellation conditions \( C(\varepsilon_n, \mu_n, \rho_n) \) over groups

\[
G_n = \left\langle S \mid \bigcup_{i=0}^{n} R_i \right\rangle.
\]

Hyperbolicity of \( G_n \) is derived for hyperbolicity of \( G_0 \) by induction).

Moreover, the larger is \( n \), the stronger is the condition \( C(\varepsilon_n, \mu_n, \rho_n) \). Parameters \( \alpha \) and \( K \) impose some uniform restrictions on \( \varepsilon_n, \mu_n, \rho_n \).
Ingredients of the proof

2. If a group $G$ admits a graded presentation satisfying $Q(\alpha, K)$ for some specific values of the parameters, then $G$ is strongly lacunary hyperbolic.

3. Previously known constructions of infinite periodic groups and Tarskii Monsters (due to Olshanskii) can be improved so that the resulting groups satisfy the small cancellation condition $Q(\alpha, K)$ for the values of the parameters we need.
PART II

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Lacunary hyperbolic groups
Equivalent definitions of lacunary hyperbolic groups

Theorem (Olshanskii–Osin–Sapir)

Let $G$ be a finitely generated group. Then the following conditions are equivalent.

1. At least one asymptotic cone of $G$ is an $\mathbb{R}$-tree.
2. There exists a scaling sequence $d = (d_n)$ such that $\text{Con}^\omega (G, d)$ is an $\mathbb{R}$-tree for any ultrafilter $\omega$.
3. $G$ is the direct limit of a sequence of hyperbolic groups $G_i = \langle S_i \rangle$, $\# S_i < \infty$, and epimorphisms

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} \ldots,$$

where $\alpha_i(S_i) = S_{i+1}$, $\alpha_i$ is injective on balls of radius $r_i$, and $G_i$ is $o(r_i)$-hyperbolic with respect to $S_i$.

Definition

If a finitely generated group satisfies these equivalent properties, it is called lacunary hyperbolic.
AMENABLE LACUNARY HYPERBOLIC GROUPS
Problem (Kleiner)

Suppose that a group $G$ is finitely generated, amenable, and not virtually cyclic. Can it have cut points in at least one asymptotic cone?

Definition

The class of *elementary amenable* groups is the smallest class of groups that contains all abelian and finite groups and is closed under taking subgroups, extensions, and directed limits.

Theorem (Olshanskii–Osin–Sapir)

There exists a finitely generated group $G$ satisfying the following properties.

1. $G$ is not virtually cyclic.
2. $G$ is lacunary hyperbolic.
3. $G$ splits as $1 \to L \to G \to \mathbb{Z} \to 1$, where $L$ is locally finite. In particular, $G$ is elementary amenable.
Construction of the group

Pick a prime $p$ and a non-decreasing sequence $c$ of positive integers $c_1 \leq c_2 \leq \ldots$. Consider the group $A(p, c)$ generated by $a_i, i \in \mathbb{Z}$ subject to the following relations:

$$a_i^p = 1, \quad i \in \mathbb{Z},$$

$$[\ldots[a_{i_0}, a_{i_1}], \ldots, a_{i_{cn}}] = 1$$

for every $n$ and all commutators with $\max_{j,k} |i_j - i_k| \leq n$.

The map $a_i \to a_{i+1}$ ($i \in \mathbb{Z}$) extends to an automorphism of $A(p, c)$. Let

$$G(p, c) = \langle A(p, c), t \mid ta_it^{-1} = a_{i+1}, \quad i \in \mathbb{Z} \rangle.$$ 

Clearly $G(p, c) = \langle t, a_0 \rangle$.

For every $p$ and every $c$, $G(p, c)$ is not virtually cyclic and is (locally finite)-by-cyclic. Moreover, it is a directed limit of a sequence

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} \ldots,$$

of virtually free groups and $IR_{\{t, a_0\}}(\alpha_n)$ can be made arbitrary large by taking sufficiently large $c_{n+1}$. Hence $G(p, c)$ is lacunary hyperbolic for every $c$ with sufficiently fast growth.
CENTRAL EXTENSIONS OF LACUNARY HYPERBOLIC GROUPS
Theorem (Mineyev)

Let $G$ be a hyperbolic group. Then the natural map $H^n_b(G, \mathbb{Z}) \to H^n(G, \mathbb{Z})$ is surjective for all $n \geq 2$.

\[ H^2_b(G, \mathbb{Z}) \to H^2(G, \mathbb{Z}) \text{ is surjective.} \]
\[ \Downarrow \text{(Gersten)} \]

Any central extension $1 \to \mathbb{Z} \to H \to G \to 1$ is quasi-isometric to $G \times \mathbb{Z}$.
\[ \Downarrow \]

For any $d = (d_n)$ and $\omega$, $\text{Con}^\omega(H, d)$ is bi-Lipschitz equivalent to $\text{Con}^\omega(G, d) \times \mathbb{R}$. 

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Lacunary hyperbolic groups
Given a product $X \times Y$ of metric spaces $X$ and $Y$, we define a metric on $X \times Y$ by the rule

$$\text{dist}_{X \times Y}((x_1, y_1), (x_2, y_2)) = \text{dist}_X(x_1, x_2) + \text{dist}_Y(y_1, y_2).$$

We write $X \sim_{\text{Lip}} Y$ if metric spaces $X$ and $Y$ are bi-Lipschitz equivalent.

**Theorem**

*Let $N$ be a central subgroup of a finitely generated group $G$. Suppose that $\text{Con}^\omega(G/N, d)$ is an $\mathbb{R}$–tree for some $d = (d_n)$ and $\omega$. Then

$$\text{Con}^\omega(G, d) \sim_{\text{Lip}} \text{Con}^\omega(N, d) \times \text{Con}^\omega(G/N, d),$$

where $\text{Con}^\omega(N, d)$ is taken with respect to the metric on $N$ induced from $G$.***
Main example

Fix an infinite presentation

\[ H = \langle a, b \mid R_1, R_2, \ldots \rangle \]

such that:

(a) The set of relations satisfies \( C'(1/24) \).

(b) Lengths \( r_i = |R_i| \) grow sufficiently fast. In particular, \( H \) is lacunary hyperbolic.

Given a sequence of integers \( k = (k_n), \ k_n \geq 2 \), consider the central extension of \( H \) defined by

\[
G(k) = \left\langle a, b \mid [R_n, a] = 1, [R_n, b] = 1, R_n^{k_n} = 1, n = 1, 2, \ldots \right\rangle \tag{1}
\]
Problem (Drutu–Sapir)

Suppose an asymptotic cone of a finitely generated group $G$ has cut points. Does every asymptotic cone of $G$ have cut points?

By a connectedness degree $c(X) \in \{0, 1, \ldots, \infty\}$ of a metric space $X$ we mean the minimal number of points whose removal disconnects $X$.

The negative answer to the above question is provided by

Theorem (Olshanskii–Osin–Sapir)

Let $G(k)$ be the group corresponding to the sequence $k_n = m \geq 2$. Then for any ultrafilter $\omega$ and any scaling sequence $d = (d_n)$, exactly one of the following possibilities occurs and both of them can be realized for suitable $\omega$ and $d$.

1. $c(\text{Con}^\omega(G(k), d)) = m$.
2. $\text{Con}^\omega(G(k), d)$ is an $\mathbb{R}$–tree.
**Theorem (Erschler–Osin)**

*Any countable group can be realized as a subgroup of $\text{Con}^\omega(G, d)$ for some $G$, $d$, and $\omega$.***

**Theorem (Drutu-Sapir)**

*For any countable groups $Q$, there exist $G$, $d$, $\omega$ such that $\pi_1(\text{Con}^\omega(G, d))$ is the free product of uncountably many copies of $Q$.***

**Problem (Gromov)**

*Can the fundamental group of an asymptotic cone of a finitely generated group be countable and non-trivial?*

The main difficulty comes from the fact that the (uncountable) group $\prod^\omega G$ acts on $\text{Con}^\omega(G, d)$ transitively.
Let $G = G(k)$ be the group corresponding to a sequence $k = (k_n)$ such that

$$k_n \to \infty \quad \text{and} \quad k_n|R_n| = o(|R_{n+1}|).$$

Let $N = \langle R_1, R_2, \ldots \rangle$. Clearly $N$ is central in $G$.

**Theorem (Olshanskii–Osin–Sapir)**

*There exists a scaling sequence $d = (d_n)$ such that for any ultrafilter $\omega$ the following conditions hold:*

1. $\text{Con}^\omega(G/N, d)$ is an $\mathbb{R}$–tree.
2. $\text{Con}^\omega(N, d)$ is isometric to $S^1$.

*In particular,*

$$\text{Con}^\omega(G, d) \sim_{\text{Lip}} S^1 \times (\mathbb{R}–\text{tree})$$

*and* $\pi_1(\text{Con}^\omega(G, d)) = \mathbb{Z}$. 
DIVERGENCE OF GEODESICS
Definition

Let $G$ be a finitely generated 1-ended group. Let $a, b \in G$, $r = \min\{|a|, |b|\}$. Fix $0 < \delta < 1$ and define $\text{div}_G(a, b)$ as the infimum of lengths of paths connecting $a, b$ in the Cayley graph of $G$ and avoiding the ball $\text{Ball}(1, \delta r)$. The divergence function is defined by

$$\text{div}_G(n, \delta) = \sup\{\text{div}_G(a, b) \mid |a|, |b| \leq n\}.$$ 

Observation. (Drutu–Sapir) $\text{div}_G(n, \delta)$ is linear for some $\delta \in (0, 1)$ iff no asymptotic cone of $G$ has cut points.

Examples.

1. $\text{div}_{\mathbb{Z}^k}(n, \delta) \sim n$ for all $k \geq 2$ and any $\delta$.
2. (Short et al) For any non-elementary hyperbolic group $G$ and any $\delta$, $\text{div}_G(n, \delta)$ is at least exponential.
3. (Behrstock) For all but finitely many mapping class groups, the divergence function is quadratic for any $\delta$. 

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Lacunary hyperbolic groups
Problem (Behrstock)

Can the divergence function of a finitely generated group be subquadratic, but not linear?

Theorem

Let \( f : \mathbb{N} \to \mathbb{N} \) be a function such that
- \( f(n)/n \) is non-decreasing.
- \( \lim_{n \to \infty} f(n)/n = \infty \).

Then there is group \( G \) and \( \delta > 0 \) such that
- \( \text{div}_G(n, \delta)/f(n) \) is bounded.
- \( G \) is lacunary hyperbolic. In particular, \( \text{div}_G(n, \delta) \) is not linear.
Open problems

1. Is it true that the growth of every non-elementary lacunary hyperbolic group is exponential?

2. Suppose that the word problem in a lacunary hyperbolic group $G$ is decidable. Does it follows that the conjugacy problem is decidable as well?

3. What can be said about linear lacunary hyperbolic groups? Are they hyperbolic?

4. Does there exist a finitely presented group all of whose asymptotic cones are locally isometric, but not all of them are isometric?

5. Is there a finitely generated amenable non–virtually cyclic group all of whose asymptotic cones a) have cut–points; b) are locally isometric to $\mathbb{R}$–trees?