A Bound for the Order of Derivatives in the Rosenfeld-Gröbner Algorithm

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Outline

• Introduction:
  • Jacobi bound for ODE systems
  • Ritt’s proof of the Jacobi bound for linear systems
  • Motivation for our bound on the order of derivatives
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• Bound for the special case of 2 variables
Outline

- **Introduction:**
  - Jacobi bound for ODE systems
  - Ritt’s proof of the Jacobi bound for linear systems
  - Motivation for our bound on the order of derivatives
- **Bound for the special case of 2 variables**
- **General case:** $n$ variables
  - What if the set of leading variables is fixed?
  - How can the set of leading variables change?
  - Weak d-triangular sets (E. Hubert’s modification of the Rosenfeld-Gröbner algorithm)
  - Algebraic reduction w.r.t. a weak d-triangular set preserving the bound
  - Final algorithm and proof of the bound
Notation

• $\mathbb{K}$ is an ordinary differential field of characteristic zero with derivation $\delta : \mathbb{K} \to \mathbb{K}$:

$$\delta(a + b) = \delta(a) + \delta(b), \quad \delta(ab) = \delta(a)b + a\delta(b).$$

• $Y = \{y_1, \ldots, y_n\}$ is a set of differential indeterminates.

• $\delta^\infty Y = \{\delta^m y \mid y \in Y, \ m = 0, 1, 2, \ldots\}$ is the set of derivatives.

• $\mathbb{K}\{Y\} = \mathbb{K}[\delta^\infty Y]$ endowed with $\delta : \mathbb{K}\{Y\} \to \mathbb{K}\{Y\}$ is the differential ring of differential polynomials.
Jacobi bound for linear systems

- Given a system of $n$ linear differential polynomials

$$L_1, \ldots, L_n \in \mathbb{K}\{Y\}$$

which for every $y \in Y$ implies an equation in $y$ alone.

- Let $a_{ij} = \text{ord}_{y_j} L_i$, $1 \leq i, j \leq n$

  (here we assume that $\text{ord}_y f = -\infty$ if $f$ does not involve any derivatives of $y$)

- For a permutation $\pi \in S_n$, let

$$d_\pi = a_{1\pi(1)} + \cdots + a_{n\pi(n)}$$

be called a diagonal sum.

- Let $h = \max_{\pi \in S_n} d_\pi$. 
Jacobi bound for linear systems

**Theorem** [Ritt, 1935] There exists a triangular system of differential polynomials $R_1, \ldots, R_n$ equivalent to $L_1, \ldots, L_n$ and satisfying $\sum_{i=1}^{n} \text{ord}_{y_i} R_i \leq h$.

**Proof...**

- Show that there exists a finite diagonal sum.
- Consider elimination ranking $y_1 > \ldots > y_n$.
- If $a_{i1}$ participates in a maximum diagonal sum, then reduction w.r.t. $L_i$, if it is possible, does not increase $h$. 
Jacobi bound for linear systems

Theorem [Ritt, 1935] There exists a triangular system of differential polynomials $R_1, \ldots, R_n$ equivalent to $L_1, \ldots, L_n$ and satisfying
\[ \sum_{i=1}^{n} \text{ord}_{y_i} R_i \leq h. \]

Proof...
If such reductions are not possible, this is because

- Only one $L_i$ involves $y_1 \Rightarrow$ proceed similarly with the elimination of $y_2, \ldots, y_{n-1}$.
- There exists $i$ such that $a_{i1}$ is maximal among $a_{11}, \ldots, a_{n1}$ and participates in a finite diagonal sum. Without loss of generality, assume that $i = n$. 
Jacobi bound for linear systems

Theorem [Ritt, 1935] There exists a triangular system of differential polynomials $R_1, \ldots, R_n$ equivalent to $L_1, \ldots, L_n$ and satisfying $\sum_{i=1}^{n} \text{ord}_{y_i} R_i \leq h$.

Proof...

- Change the indices of $L_1, \ldots, L_{n-1}$ and $y_2, \ldots, y_n$ so that

$$a_{11} + \ldots + a_{n-1,n-1}$$

is maximal. This sum is finite.

- Then one can reduce $L_n$ w.r.t. $L_1$ without increasing $h$. 

$\square$
Notation

- Fix a ranking <: a total order on derivatives such that for all $u, v \in \delta^\infty Y$

  \[ u < \delta u \] and \[ u < v \Rightarrow \delta u < \delta v \].
Notation

• Fix a ranking $\prec$: a total order on derivatives such that for all $u, v \in \delta^\infty Y$

$$[u < \delta u] \text{ and } [u < v \Rightarrow \delta u < \delta v].$$

• For a polynomial $f$, let $u_f = \delta^k y_i$ be the derivative of the highest rank w.r.t. $\leq$ occurring in $f$. Then

$$f = i_f u_f^d + g(u_f), \quad \deg g < d.$$
Notation

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$$f = i_f u_f^d + g(u_f), \text{ deg } g < d.$$  

• $lv f = y_i$, $ld f = u_f$, $rk f = u_f^d$, $s_f = i_{\delta f}.$
Notation

• Fix a ranking $\prec$: a total order on derivatives such that for all $u, v \in \delta^\infty Y$

$$[u < \delta u] \text{ and } [u < v \Rightarrow \delta u < \delta v].$$

• For a polynomial $f$, let $u_f = \delta^k y_i$ be the derivative of the highest rank w.r.t. $\leq$ occurring in $f$. Then

$$f = i_f u_f^d + g(u_f), \quad \deg g < d.$$

• lv $f = y_i$, ld $f = u_f$, rk $f = u_f^d$, sf $= i_\delta f$.

• Ranks $u_1^d$ and $u_2^d$ can be compared w.r.t. $\prec$:

$$u_1^d < u_2^d \iff [u_1 < u_2] \text{ or } [u_1 = u_2 \text{ and } d_1 < d_2].$$
Some basics of differential algebra

- Polynomial $f$ is \textit{algebraically reduced} w.r.t. $g$, if $\deg_{u_g} f < \deg_{u_g} g$.

- $f$ is \textit{partially reduced} w.r.t. $g$, if $f$ is free of $\delta^k u_g$, $k > 0$.

- $f$ is (fully) \textit{reduced} w.r.t. $g$, if $f$ is algebraically and partially reduced w.r.t. $g$.

- Set $A$ is \textit{autoreduced}, if every element of $A$ is reduced w.r.t. every other element of $A$.

- For an autoreduced set $A$, let $\min A$ denote the polynomial in $A$ of the least rank.

- For autoreduced sets $A$ and $B$, $\text{rk} A < \text{rk} B$ iff

$$[\text{rk} B \subset \text{rk} A] \text{ or } [\min(\text{rk} A \setminus \text{rk} B) < \min(\text{rk} B \setminus \text{rk} A)].$$
Regular ideals

- For any finite polynomial sets $A, H$, ideal

$$[A] : H^\infty = \{f \mid \exists h \in H^\infty \, hf \in [A]\}$$

is differential.

- Ideal $[A] : H^\infty$ is called regular, if
  - $A$ is autoreduced
  - $H \supset H_A = \{i_f, s_f \mid f \in A\}$
  - $H$ is partially reduced w.r.t. $A$.

- Theorem. [Boulier et al, 1995] Regular ideals are radical.

- Rosnefeld’s Lemma. If differential ideal $[A] : H^\infty$ is regular and polynomial $f$ is partially reduced w.r.t. $A$, then

$$f \in [A] : H^\infty \iff f \in (A) : H^\infty$$
Regular decomposition

- The Rosenfeld-Gröbner algorithm yields a regular decomposition of a radical differential ideal:

\[ \{ F \} = \bigcap_{i=1}^{k} R_i, \quad R_i = [A_i] : H_i^\infty. \]

- There exist efficient algebraic methods (plus parallel and modular Monte-Carlo algorithms currently under development by M. Moreno Maza et al) for computing a regular decomposition of a radical ideal:

\[ \sqrt{G} = \bigcap_{i=1}^{l} J_i, \quad J_i = (A_i) : H_i^\infty. \]
Motivation for our bound

• Given a system of differential polynomials $F$, find a number $d$, so that every algebraic regular decomposition of the radical algebraic ideal

$$\sqrt{F(d)}, \quad F(d) = \{ f^{(i)} \mid f \in F, \ 0 \leq i \leq d \}$$

“yields” a regular decomposition of $\{F\}$.

• First step: estimate the order of differential polynomials in a regular decomposition

$$\{ F \} = \bigcap_{i=1}^{k} [A_i] : H_i^\infty.$$
Rosenfeld-Gröbner algorithm

Algorithm Rosenfeld-Gröbner($F_0$)

Input: A finite set of differential polynomials $F_0$

Output: A finite set $T$ of regular systems such that $\{F_0\} = \bigcap_{(A,H)\in T} [A] : H^\infty$

$T := \emptyset$

$U := \{(F_0, \emptyset)\}$

while $U \neq \emptyset$ do

Take and remove any $(F, H) \in U$

Let $C$ be an autoreduced subset of $F$ of the least rank

$R := d\text{-rem}(F \setminus C, C) \setminus \{0\}$

if $R = \emptyset$ then

if $1 \notin (C) : (d\text{-rem}(H, C) \cup H_C)^\infty$ then $T := T \cup \{(C, d\text{-rem}(H, C) \cup H_C)\}$

else

$U := U \cup \{(C \cup R, H \cup H_C)\}$

end if

$U := U \cup \{(F \cup \{h\}, H) \mid h \in H_C, h \in \mathbb{K}\}$

end if

end while

return $T$
Special case: $n = 2$

- Let $F \subset \mathbb{K}\{y, z\}$.
- Let $m_y(F)$ and $m_z(F)$ be the maximal orders of derivatives of $y$ and $z$ occurring in $F$.
- Let $M(F) = m_y(F) + m_z(F)$.

**Lemma.** For all $(F, H) \in U$ in the Rosenfeld-Gröbner algorithm,

$$M(F) \leq M(F_0).$$

**Proof...**
Show that $M(F)$ cannot increase in the Rosenfeld-Gröbner algorithm:

- Let $(F, H) \in U$.
- Let $C$ be an autoreduced subset of $F$ of the least rank.
Special case: \( n = 2 \)

- Let \( F \subset \mathbb{K}\{y, z\} \).
- Let \( m_y(F) \) and \( m_z(F) \) be the the maximal orders of derivatives of \( y \) and \( z \) occurring in \( F \).
- Let \( M(F) = m_y(F) + m_z(F) \).

**Lemma.** For all \( (F, H) \in U \) in the Rosenfeld-Gröbner algorithm,

\[
M(F) \leq M(F_0).
\]

**Proof...**

- \( |C'| \leq 2 \).
- Let \( R = \text{d-rem}(F \setminus C, C) \).
Special case: $n = 2$

- Let $F \subset \mathbb{K}\{y, z\}$.

- Let $m_y(F)$ and $m_z(F)$ be the maximal orders of derivatives of $y$ and $z$ occurring in $F$.

- Let $M(F) = m_y(F) + m_z(F)$.

**Lemma.** For all $(F, H) \in U$ in the Rosenfeld-Gröbner algorithm,

$$M(F) \leq M(F_0).$$

**Proof...**

- Let $|C| = 1$. Without loss of generality, $\text{ld } C = \{y^{(d_y)}\}$.

- $m_y(C \cup R) = d_y$, $m_z(C \cup R) \leq m_z(F) + (m_y(F) - d_y)$.

- Therefore $M(C \cup R) \leq M(F)$. 
Special case: \( n = 2 \)

- Let \( F \subset \mathbb{K}\{y, z\} \).
- Let \( m_y(F) \) and \( m_z(F) \) be the the maximal orders of derivatives of \( y \) and \( z \) occurring in \( F \).
- Let \( M(F) = m_y(F) + m_z(F) \).

**Lemma.** For all \((F, H) \in U\) in the Rosenfeld-Gröbner algorithm,

\[
M(F) \leq M(F_0).
\]

**Proof...**

- Let \(|C| = 2\). Then \( \text{ld } C = \{y^{(d_y)}, z^{(d_z)}\} \) and

\[
M(C \cup R) = d_y + d_z \leq M(F).
\]
Special case: \( n = 2 \)

- Let \( F \subset \mathbb{K}\{y, z\} \).
- Let \( m_y(F) \) and \( m_z(F) \) be the maximal orders of derivatives of \( y \) and \( z \) occurring in \( F \).
- Let \( M(F) = m_y(F) + m_z(F) \).

Lemma. For all \((F, H) \in U\) in the Rosenfeld-Gröbner algorithm,

\[ M(F) \leq M(F_0). \]

Proof...

- Finally, if \( G \subset F \cup H_F \), then \( M(G) \leq M(F) \).

\( \square \)
General case; fixed leading variables

- Let $F \subset \mathbb{K}\{y_1, \ldots, y_n\}$.
- Let $C$ be an autoreduced subset of $F$ of the least rank with
  \[ \text{ld } C = \{y_1^{(d_1)}, \ldots, y_k^{(d_k)}\}. \]
- Then

\[
m_i(C \cup R) \leq \begin{cases} 
  d_i, & i = 1, \ldots, k \\
  m_i(F) + \max_{1 \leq j \leq k} (m_j(F) - d_j), & i = k + 1, \ldots, n
\end{cases}
\]
General case; fixed leading variables

Define

\[ M_{1v}C(F') = M_{y_1,...,y_k}(F') = (n - k) \sum_{i=1}^{k} m_i(F') + \sum_{i=k+1}^{n} m_i(F') \quad (1 \leq |C| < n). \]

Then inequality

\[ m_i(C \cup R) \leq \left\{ \begin{array}{ll} d_i, & i = 1,\ldots,k \\ m_i(F') + \max_{1 \leq j \leq k} (m_j(F') - d_j), & i = k+1,\ldots,n \end{array} \right. \]

implies:

\[ M_{1v}C(C \cup R) = M_{y_1,...,y_k}(C \cup R) = \]

\[ (n - k) \sum_{i=1}^{k} m_i(C \cup R) + \sum_{i=k+1}^{n} m_i(C \cup R) \leq \]

\[ (n - k) \sum_{i=1}^{k} d_i + \sum_{i=k+1}^{n} m_i(F') + (n - k) \max_{1 \leq j \leq k} (m_j(F') - d_j) \leq \]

\[ (n - k) \sum_{i=1}^{k} m_i(F') + \sum_{i=k+1}^{n} m_i(F') - \]

\[ -(n - k) \sum_{i=1}^{k} (m_i(F') - d_i) + (n - k) \max_{1 \leq j \leq k} (m_j(F') - d_j) \leq M_{1v}C(F'). \]
Changing leading variables

• A non-leading variable $y_{k+1}$ becomes leading:

$$M_{y_1, \ldots, y_{k+1}}(F) = (n - k - 1) \sum_{i=1}^{k+1} m_i(F) + \sum_{i=k+2}^{n} m_i(F) \leq M_{y_1, \ldots, y_k}(F) + (n - k - 2)m_{k+1}(F) \leq (n - k - 1)M_{y_1, \ldots, y_k}(F).$$

• A leading variable becomes non-leading: make sure this does not happen!
Leading variables become non-leading

Example 1:

- $F = \{x, x^2 + z, y^2 + z\}, \ x > y > z$
- $C = \{x, y^2 + z\}, \ lv\ C = \{x, y\}$
- $R = d\text{-rem}(F \setminus C, C) = \{z\}$
- $F_1 = C \cup R = \{x, y^2 + z, z\}$
- $C_1 = \{x, z\}, \ lv\ C_1 = \{x, z\}$
- $y \in lv\ C$ but $y \notin lv\ C_1$
- $y$ disappeared from leading variables only temporarily: reduce $y^2 + z$ w.r.t. $z$, and $y$ becomes a leading variable again.
- $\Rightarrow$ one can try to replace autoreduced sets by weak $d$-triangular sets in the Rosenfeld-Gröbner algorithm
Leading variables become non-leading

Example 2:

- $F = \{x, x^2 + z, zy^2\}$, $x > y > z$
- $C = \{x, zy^2\}$, $\text{lv } C = \{x, y\}$
- $R = \text{d-rem}(F \setminus C, C) = \{z\}$
- $F_1 = C \cup R = \{x, zy^2, z\}$
- $C_1 = \{x, z\}$, $\text{lv } C_1 = \{x, z\}$
- $y$ disappeared from leading variables permanently:
  \[ zy^2 \rightarrow z 0. \]

- **Observation:** In the component $(F_1, H_1)$, where $H_1 = H \cup H_C$, we have $z \in F_1 \cap H_1$, hence
  \[ \{F_1\} : H_1^\infty = (1). \]
Differentially triangular sets

- A set of polynomials $A$ is a weak differentially triangular set, if $\text{ld } A$ is autoreduced.
- A weak differentially triangular set $A$ is differentially triangular, if every element of $A$ is partially reduced w.r.t. the other elements of $A$.
- One can expand the definition of regular ideals [Hubert]: Ideal $[A] : H^\infty$ is called regular, if
  - $A$ is differentially triangular
  - $H \supset s_A$
  - $H$ is partially reduced w.r.t. $A$. 
Modified Rosenfeld-Gröbner algorithm

**Algorithm** Rosenfeld-Gröbner($F_0$) (based on [Hubert, 2001])

*Input:* A finite set of differential polynomials $F_0$

*Output:* A finite set $T$ of regular systems such that $\{F_0\} = \bigcap_{(A,H) \in T} [A : H^\infty}$

1. $T := \emptyset$
2. $U := \{(F_0 \setminus \{\min F_0\}, \{\min F_0\}, \emptyset)\}$
3. while $U \neq \emptyset$ do
   4. Take and remove any $(F, C, H) \in U$
   5. $R := \operatorname{d-rem}(F, C) \setminus \{0\}$
   6. if $R = \emptyset$ then $T := T \cup \operatorname{Autoreduce&Check}(C, H \cup H_C)$
   7. else $C^\succ := \{p \in C \mid \operatorname{lv} p = \operatorname{lv}(\min R)\}$
      8. $\bar{C} := C \setminus C^\succ \cup \{\min R\} \quad \# \text{ Note: } \bar{C} \text{ is a weak d-triangular set s.t.}$
      9. $\bar{F} := C^\succ \cup R \setminus \{\min R\} \quad \# \quad \operatorname{rk} \bar{C} < \operatorname{rk} C \text{ and } \operatorname{lv} C \subseteq \operatorname{lv} \bar{C}$
      10. $\bar{H} := \operatorname{d-rem}(H \cup H_{\bar{C}}, \bar{C})$
      11. if $0 \notin \bar{H}$ then $U := U \cup \{(\bar{F}, \bar{C}, \bar{H})\}$
   12. end if
   13. $U := U \cup \{(F \cup \{h\}, C, H) \mid h \in H_C, \ h \notin \mathbb{K}\}$
14. end while
15. return $T$
Reduction w.r.t. a weak d-Δ set

Algorithm Rosenfeld-Gröbner($F_0$)

Input: A finite set of differential polynomials $F_0$

Output: A finite set $T$ of regular systems such that $\{F_0\} = \bigcap_{(A,H) \in T} [A] : H^\infty$

... while $U \neq \emptyset$ do
  Take and remove any $(F,C,H) \in U$
  Let $m_i = \max \{\text{ord}_{y_i} f \mid f \in F \cup C\}, i = 1, \ldots, n$
  $B := \text{Differentiate&Autoreduce}(C, \{m_i\}_{i=1}^n)$
  if $B \neq \emptyset$ then
    $R := \text{alg-rem}(F, B) \setminus \{0\}$
    if $R = \emptyset$ then $T := T \cup \text{Autoreduce&Check}(C,H \cup H_C)$
    else
      ...
    end if
  end if
  $U := U \cup \{(F \cup \{h\}, C, H) \mid h \in H_C, h \notin \mathbb{K}\}$
end while
return $T$
Algorithm Differentiate&Autoreduce

Algorithm Differentiate&Autoreduce($C, \{m_i\}$)

Input: a weak d-triangular set $C = C_1, \ldots, C_k$ with $\text{ld } C = y_1^{(d_1)}, \ldots, y_k^{(d_k)}$, and a set of non-negative integers $\{m_i\}_{i=1}^n$, $m_i \geq m_i(C)$

Output: set $B = \{B_i^j \mid 1 \leq i \leq k, 0 \leq j \leq m_i - d_i\}$ satisfying

$B \subset [C], \text{ rk } B_i^0 = \text{rk } C_i, \text{ rk } B_i^j = y_i^{(d_i+j)} (j > 0)$

$i_{B_i^j} \in H^\infty_C + [C] (j \geq 0)$

$B_i^j$ is partially reduced w.r.t. $C \setminus \{C_i\}$

$m_i(B) \leq m_i + \sum_{j=1}^k (m_j - d_j), i = k + 1, \ldots, n$

or $\emptyset$, if it is detected that $[C] : H^\infty_C = (1)$

for $i := 1$ to $k$

$B_i^0 := \text{alg-rem}(C_i, \{B_l^r \mid 1 \leq l < i, 0 < r \leq m_l - d_l\})$

if $\text{rk } B_i^0 \neq \text{rk } C_i$ then return $\emptyset$

for $j := 1$ to $m_i - d_i$

$B_i^j := \text{alg-rem}(\delta B_i^{j-1}, \delta(C \setminus \{C_i\}))$

if $\text{ld } B_i^j \neq y_i^{(d_i+j)}$ then return $\emptyset$

end for

end for

return $B$
Lemma. Let $C$ be a weak d-triangular set, and let $f$ be a polynomial such that $lv f \not\in lv C$ and $i_f \in H_C^\infty + [C]$. Let $f \rightarrow_C g$. Then

- $rk g \neq rk f \Rightarrow [C] : H_C^\infty = (1)$
- $rk g = rk f \Rightarrow i_g \in H_C^\infty + [C]$

- $B_1^0 = C_1$ is partially reduced w.r.t. $C_2, \ldots, C_k$.
- $\delta B_1^0$ is reduced w.r.t. $\delta^l C_i$, $l > 1$, $i = 2, \ldots, k$
- $rk \delta B_1^0 = y_1^{(d_1+1)}$
- $B_1^1 = alg-rem(\delta B_1^0, \delta(C \setminus \{C\})$
- Lemma $\Rightarrow [C] : H_C^\infty = (1)$ or

$$rk B_1^1 = y_1^{(d_1+1)} \text{ and } i_{B_1^1} \in H_C^\infty + [C].$$
Lemma. Let $C$ be a weak $d$-triangular set, and let $f$ be a polynomial such that $\text{lv } f \not\in \text{lv } C$ and $i_f \in H_C^\infty + [C]$. Let $f \rightarrow_C g$. Then

- $\text{rk } g \neq \text{rk } f \Rightarrow [C] : H_C^\infty = (1)$
- $\text{rk } g = \text{rk } f \Rightarrow i_g \in H_C^\infty + [C]

- $B_1^1$ is partially reduced w.r.t. $C_2, \ldots, C_k$.
- $\Rightarrow$ similarly for all $B_1^r$, $1 < r < m_1 - d_1$.
- For $B_1 = B_1^0, \ldots, B_1^{m_1-d_1}$, we have:

$$B_1 \subset [C], \quad H_{B_1} \subset H_C^\infty + [C]$$
Lemma. Let $C$ be a weak $d$-triangular set, and let $f$ be a polynomial such that $\text{lv} f \notin \text{lv} C$ and $i_f \in H_C^\infty + [C]$. Let $f \rightarrow_C g$. Then

- $\text{rk } g \neq \text{rk } f \Rightarrow [C] : H_C^\infty = (1)$
- $\text{rk } g = \text{rk } f \Rightarrow i_g \in H_C^\infty + [C]$

- $B_2^0 = \text{alg-rem}(C_2, \{B_1^0, \ldots, B_1^{m_1-d_1}\})$

- $C_2$ is partially reduced w.r.t. $C_3, \ldots, C_k$ and $y_1^{(d_1+l)}$, $l > m_1 - d_1$.

- By Lemma, two cases are possible:
  - $\text{rk } B_2^0 = \text{rk } C_2$, $i_{B_2^0} \in H_B^\infty + [B] \subset H_C^\infty + [C]$
  - $[B] : H_B^\infty = (1) \Rightarrow [C] : H_C^\infty = (1)$

- Similarly for $B_2^1, \ldots, B_2^{m_2-d_2}$ and $B_i^r$, $i > 2$. 
Inequality

\[ m_i(B) \leq m_i + \sum_{j=1}^{k} (m_j - d_j), \quad i = k + 1, \ldots, n \]

follows from the fact that the two nested loops

\begin{verbatim}
for i := 1 to k do
    ...
    for j := 1 to m_i - d_i do
        ...
end for
end for
\end{verbatim}

have \( \sum_{j=1}^{k} (m_j - d_j) \) iterations, and at each iteration each polynomial is differentiated at most once.
Algorithm Rosenfeld-Gröbner($F_0$)

Output: $\{F_0\} = \bigcap_{(A,H) \in T}[A] : H^\infty$ satisfying $M(A) \leq (n-1)!M(F_0)$, $(A,H) \in T$

$T := \emptyset$, $U := \{(F_0 \setminus \{\min F_0\}, \{\min F_0\}, \emptyset)\}$

while $U \neq \emptyset$ do

Take and remove any $(F, C, H) \in U$

Let $m_i = \max\{\operatorname{ord}_{y_i} f \mid f \in F \cup C\}$, $i = 1, \ldots, n$

$B := \text{Differentiate} \& \text{Autoreduce}(C, \{m_i\}_{i=1}^n)$

if $B \neq \emptyset$ then

$R := \text{alg-rem}(F, B) \setminus \{0\}$

if $R = \emptyset$ then $T := T \cup \text{Autoreduce} \& \text{Check}(C, H \cup H_C)$

else $C^> := \{p \in C \mid \operatorname{lv} p = \operatorname{lv}(\min R)\}$

$\bar{C} := C \setminus C^> \cup \{\min R\}$

$\bar{F} := C^> \cup R \setminus \{\min R\}$

$\bar{H} := \text{d-rem}(H \cup H_{C^>}, \bar{C})$

if $0 \notin \bar{H}$ then $U := U \cup \{(\bar{F}, \bar{C}, \bar{H})\}$

end if

$U := U \cup \{(F \cup \{h\}, C, H) \mid h \in H_C, h \notin \mathbb{K}\}$

end if

end while

return $T$
Final proof of the bound

- $m_i(B) \leq m_i + \sum_{j=1}^{k} (m_j - d_j), \; k < i \leq n$

- $m_i(R) \leq \begin{cases} d_i, & 1 \leq i \leq k \\ m_i(B), & k < i \leq n \end{cases}$

- $M_{1v} C(R) \leq (n - k) \sum_{i=1}^{k} d_i + \sum_{i=k+1}^{n} m_i + \sum_{i=k+1}^{n} (m_i - d_i) \leq M_{1v} C(F \cup C)$

- Two cases are possible:
  - $|C| < n$: Again two cases:
    - $|\bar{C}| < n$:
      - $|\bar{C}| = n$: $M(\bar{F} \cup \bar{C}) = M_{1v} C(\bar{F} \cup \bar{C}) \leq M(F \cup C)$.
      - $|C| = n$. Then also $|\bar{C}| = n$ and $M(\bar{F} \cup \bar{C}) \leq \sum_{i=1}^{n} d_i \leq M(F \cup C)$.
    - Therefore $M(A) \leq (n - 1)!M(F_0)$. 

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References


