Differential Groups and Differential Relations

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Theorem: (Hölder, 1887) The Gamma function $\Gamma(x + 1) = x\Gamma(x)$ satisfies no polynomial differential equation.

Goal: Prove this using differential algebraic groups and generalize

Ex. If $y_1(x)$ and $y_2(x)$ are lin. indep. solutions of

$$y(x + 2) - xy(x + 1) + y(x) = 0$$

then $y_1(x)$, $y_1(x + 1)$ and $y_2(x)$ satisfy no polynomial differential equation.
• To study an object $\mathcal{X}$, study its group of symmetries $\mathcal{G}$

• The size of $\mathcal{G}$, measures the size of $\mathcal{X}$

• The relations defining $\mathcal{G}$ give us the relations on $\mathcal{X}$. 
• Galois Theory of Polynomial Equations
• Galois Theory of Difference Equations
• Linear Differential Algebraic Groups
• Differential Galois Theory of Difference Equations
• Differential Relations Among Solutions of Linear Difference Equations
• Final Comments
Galois Theory of Polynomial Equations

\[ f(y) = 0, \ f \in k[y] \text{ of degree } n \text{ and irreducible} \]

Galois group = the group of transformations of the roots of \( f \) that preserve all algebraic relations among them.

More formally:

\text{Splitting Ring: } K = k[y_1, \ldots, y_n, (\prod_{i<j}(y_i - y_j))^{-1}]/M = k[\alpha_1, \ldots, \alpha_n], \quad M \text{ a max ideal containing } (f(y_1), \ldots, f(y_n))

\text{Note: } K \text{ is a field and all such are isomorphic.}

Galois group = \( \text{Gal}(K/k) = \{ \sigma : K \to K \mid \sigma \text{ is a } k\text{-autom.} \} \)
\[ K = k[\alpha_1, \ldots, \alpha_n] \]
\[ \alpha = (\alpha_1, \ldots, \alpha_n), \quad V = \{ \sigma(\alpha) \mid \sigma \in \text{Gal}(K/k) \} \subset K^n \]

- \( V \) is a variety, inv. under \( \text{Gal}(\bar{k}/k) \) \( \Rightarrow \) \( V \) defined over \( k \)

\( \text{Gal}(K/k) \) acts trans. and freely on \( V \) \( \Rightarrow \) \( V \) is a \( \text{Gal}(K/k) \)-torsor

- \( K = k[\alpha_1, \ldots, \alpha_n] = \) coordinate ring of \( V \)

\[ K^{\text{Gal}(K/k)} = k \quad |\text{Gal}(K/k)| = |V| = \dim_k K \]

The size of \( \text{Gal}(K/k) \) measures relations among the roots.

**Ex.** \( |\text{Gal}(K/k)| = \deg(f) \Rightarrow \) all roots are expressed in terms of one.
Galois Theory of Difference Equations

$k$ - field, $\sigma$ - an automorphism  \[ \text{Ex. } \mathbb{C}(x), \; \sigma(x) = x + 1, \; \sigma(x) = qx \]

Difference Equation: \[ \sigma(Y) = AY \; A \in \text{GL}_n(k) \]

Splitting Ring: \[ k[Y, \frac{1}{\det(Y)}], \; Y = (y_{i,j}) \text{ indeterminates} \; , \sigma(Y) = AY, \]
\[ M = \text{max } \sigma\text{-ideal} \]
\[ R = k[Y, \frac{1}{\det(Y)}]/M = k[Z, \frac{1}{\det(Z)}] = \sigma\text{-Picard-Vessiot Ring} \]

- $M$ is radical $\Rightarrow$ $R$ is reduced

- If $C = k^\sigma = \{ c \in k \mid \sigma c = c \}$ is alg closed $\Rightarrow$ $R$ is unique and $R^\sigma = C$

Ex.

\[ k = \mathbb{C} \; \sigma(y) = -y \]
\[ R = \mathbb{C}[y, \frac{1}{y}]/(y^2 - 1) \]
σ-Galois Group: $\text{Gal}_\sigma(R/k) = \{ \phi : R \to R \mid \phi \text{ is a } \sigma \text{ } k\text{-automorphism} \}$

Ex.

$$k = \mathbb{C} \quad \sigma(y) = -y \Rightarrow R = \mathbb{C}[y, \frac{1}{y}]/(y^2 - 1)$$

$$\text{Gal}_\sigma(R/k) = \mathbb{Z}/2\mathbb{Z}$$

Ex.

$$k = \mathbb{C}(x), \sigma(x) = x + 1$$

$$\sigma^2 y - x\sigma y + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y$$

$$R = k[Y, \frac{1}{\det(Y)}]/(\det(Y) - 1), \text{ Gal}_\sigma = \text{SL}_2(\mathbb{C})$$

Ex.

$$\sigma(y) - y = f, \ f \in k \Leftrightarrow \sigma \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

$$\phi \in \text{Gal}_\sigma \Rightarrow \phi(y) = y + c_\phi, c_\phi \in C$$

$$\text{Gal}_\sigma = (C, +) \text{ or } \{0\}$$
\begin{itemize}
  \item $\phi \in \text{Gal}_\sigma$, $\sigma(Z) = AZ \Rightarrow \phi(Z) = Z[\phi]$, $[\phi] \in \text{GL}_n(C)$
  \begin{align*}
    \text{Gal}_\sigma \hookrightarrow \text{GL}_n(C) & \text{ and the image is Zariski closed} \\
    \text{Gal}_\sigma = G(C), G \text{ a lin. alg. gp. }/C.
  \end{align*}
  \item $R = \text{coord. ring of a } G\text{-torsor}$
  \begin{align*}
    R^{\text{Gal}_\sigma} = k \\
    \dim(G) = \text{Krull dim}_k R \ (\simeq \text{trans. deg. of quotient field})
  \end{align*}
\end{itemize}
The size of $\text{Gal}(K/k)$ measures algebraic relations among the solutions.

Ex.

$$\sigma^2y - x \sigma y + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y$$

$\text{Gal}_\sigma = \text{SL}_2(\mathbb{C})$

$$3 = \dim \text{SL}_2(\mathbb{C}) = \text{tr. deg}_k k(y_1, y_2, \sigma(y_1), \sigma(y_2))$$

$\Rightarrow y_1, y_2, \sigma(y_1)$ alg. indep. over $k$
Ex. \( f_1, \ldots, f_n \in k, \ k \) a difference field w. alg. closed const. 
\[
\sigma(y_1) - y_1 = f_1 \\
\vdots \\
\sigma(y_n) - y_n = f_n
\]
Picard-Vessiot ring = \( k[y_1, \ldots, y_n] \)

Prop. \( y_1, \ldots, y_n \) alg. dep. over \( k \)

if and only if

\( \exists g \in k \) and a const coeff. linear form \( L \) s.t. \( L(y_1, \ldots, y_n) = g \)

(equiv., \( c_1 f_1 + \ldots + c_n f_n = \sigma(g) - g \))

Proof. \( \text{Gal}_\sigma \subset (C, +)^n \).

\( \Rightarrow \) Alg. dependent \( \Rightarrow \ \text{Gal}_\sigma \subsetneq (C, +)^n \)

\( \Rightarrow \ \exists L \) s.t. \( \text{Gal}_\sigma \subset \{(c_1, \ldots, c_n) \mid L(c_1, \ldots, c_n) = 0\} \)

\( \phi \in \text{Gal}_\sigma, \ \phi(L(y_1, \ldots, y_n)) = L(y_1 + c_1, \ldots, y_n + c_n) \)

\( = L(y_1, \ldots, y_n) + L(c_1, \ldots, c_n) = L(y_1, \ldots, y_n) \)

So, \( L(y_1, \ldots, y_n) = g \in k. \)

Ex. \( y(x + 1) - y(x) = \frac{1}{x} \Rightarrow y(x) \) is not alg. over \( \mathbb{C}(x) \).
Linear Differential Algebraic Groups

P. Cassidy-“Differential Algebraic Groups” Am. J. Math. 94(1972),891-954
+ 5 more papers, book by Kolchin, papers by Buium, Pillay et al., Ovchinnikov

$(k, \delta) = \text{a differentially closed differential field}$

**Definition:** A subgroup $G \subset GL_n(k) \subset k^{n^2}$ is a **linear differential algebraic group** if it is Kolchin-closed in $GL_n(k)$, that is, $G$ is the set of zeros in $GL_n(k)$ of a collection of differential polynomials in $n^2$ variables.

**Ex.** Any linear algebraic group defined over $k$, that is, a subgroup of $GL_n(k)$ defined by (algebraic) polynomials, e.g., $GL_n(k), SL_n(k)$

**Ex.** Let $C = \ker \delta$ and let $G(k)$ be a linear algebraic group defined over $k$. Then $G(C)$ is a linear **differential** algebraic group (just add $\{\delta y_{i,j} = 0\}_{i,j=1}^n$ to the defining equations!)
Ex. Differential subgroups of $G_a(k) = (k, +) = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \right\} \mid z \in k$}

The linear differential subgroups are all of the form

$$G^L_a = \{ z \in k \mid L(z) = 0 \}$$

where $L$ is a linear homogeneous differential polynomial.

For example, if $m = 1$,

$$G^\delta_a = \{ z \in k \mid \delta(z) = 0 \} = G_a(C)$$

Ex. Differential subgroups of $G^n_a(k) = (k^n, +)$

The linear differential subgroups are all of the form

$$G^L_a = \{ (z_1, \ldots, z_n) \in k^n \mid L(z_1, \ldots, z_n) = 0 \}$$

where $L$ is a linear homogeneous differential polynomial.
Ex. \(H\) a Zariski-dense proper differential subgroup of \(\text{SL}_n(k)\)

\[\Rightarrow \exists g \in \text{SL}_n(k)\text{ such that } gHg^{-1} = \text{SL}_n(C), \ C = \ker(\delta).\]

In general if \(H\) a Zariski-dense proper differential subgroup of \(G \subset \text{GL}_n(k)\), a simple algebraic group defined over \(C\)

\[\Rightarrow \exists g \in \text{GL}_n(k)\text{ such that } gHg^{-1} = G(C), \ C = \ker(\delta).\]
Differential Galois Theory of Difference Equations

\( k \) - field, \( \sigma \) - an automorphism \( \delta \) - a derivation s.t. \( \sigma \delta = \delta \sigma \)

**Ex.** \( \mathbb{C}(x) \) : \( \sigma(x) = x + 1, \delta = \frac{d}{dx} \)
\( \sigma(x) = qx, \delta = x \frac{d}{dx} \)
\( \mathbb{C}(x,t) \) : \( \sigma(x) = x + 1, \delta = \frac{\partial}{\partial t} \)

Difference Equation: \( \sigma(Y) = AY, A \in \text{GL}_n(k) \)

Splitting Ring: \( k\{Y, \frac{1}{\det(Y)}\} = k[Y, \delta Y, \delta^2 Y, \ldots, \frac{1}{\det(Y)}] \)

\( Y = (y_{i,j}) \text{ differential indeterminates} \)
\( \sigma(Y) = AY, \sigma(\delta Y) = A(\delta Y) + (\delta A)Y, \ldots \)

\( M = \max \sigma \delta \)-ideal

\( R = \frac{k\{Y, \frac{1}{\det(Y)}\}}{M} = k\{Z, \frac{1}{\det(Z)}\} = \sigma \delta \)-Picard-Vessiot Ring
$k$ - $\sigma\delta$ field

$\sigma(Y) = AY, \ A \in \text{GL}_n(k)$

$R = k\{Z, \frac{1}{\det(Z)}\}$ - $\sigma\delta$-Picard-Vessiot ring

- $R$ is reduced

- If $C = k^\sigma = \{c \in k \mid \sigma c = c\}$ is differentially closed
  $\Rightarrow R$ is unique and $R^\sigma = C$
\(\sigma\delta\text{-Galois Group: } \text{Gal}_{\sigma\delta}(R/k) = \{ \phi : R \to R \mid \phi \text{ is a } \sigma\delta k\text{-automorphism} \}\)

- \(\phi \in \text{Gal}_{\sigma\delta} \Rightarrow \phi(Z) = Z[\phi], \ [\phi] \in \text{GL}_n(C)\)
  - \(\text{Gal}_{\sigma\delta} \hookrightarrow \text{GL}_n(C)\) and the image is Kolchin closed
  - \(\text{Gal}_{\sigma\delta} = G(C), G \text{ a lin. differential alg. gp. /}C\).

- \(\text{Gal}_{\sigma\delta}\) is Zariski dense in \(\text{Gal}_\sigma\)

- \(R = \text{coord. ring of a } G\text{-torsor}\)
  - \(R^{\text{Gal}_{\sigma\delta}} = k\)
  - Assume \(G\) connected. Then \(\text{diff. dim}_C(G) = \text{diff. tr. deg}_k F\)
  where \(F\) is the quotient field of \(R\).
Ex. 

\[ k = \mathbb{C} \sigma(y) = -y \Rightarrow R = k[y, \frac{1}{y}]/(y^2 - 1) \]

\[ \text{Gal}_{\sigma \delta}(R/k) = \mathbb{Z}/2\mathbb{Z} \]

Ex. 

\[ \sigma(y) - y = f, \ f \in k, \ \text{Gal}_{\sigma \delta} \subset \mathbb{G}_a \]

\[ \Rightarrow \text{Gal}_{\sigma \delta} = \{ c \in R^\sigma \mid L(c) = 0 \} \text{ for some } L \in R^\sigma[\delta]. \]

Ex. 

\[ k = \mathbb{C}(x), \sigma(x) = x + 1, \ \delta(x) = 1 \]

\[ \sigma^2 y - xy + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y \]

Will show: \[ R = k\{Y, \frac{1}{\det(Y)}\}/\{\det(Y) - 1\} \]

\[ \text{Gal}_{\delta \sigma} = \text{SL}_2(\mathbb{C}) \]
Differential Relations Among Solutions of Linear Difference Equations

Groups Measure Relations

$k - \sigma \delta$ - field, $C = k^\sigma$ differentially closed.

Differential subgroups of $G^n_a(k) = (k^n, +)$ are all of the form

$$G^L_a = \{(z_1, \ldots, z_n) \in k^n \mid L(z_1, \ldots, z_n) = 0\}$$

where $L$ is a linear homogeneous differential polynomial.

\[
\Downarrow
\]

Proposition. Let $R$ be a $\sigma \delta$-Picard-Vessiot extension of $k$ containing $z_1, \ldots, z_n$ such that

$$\sigma(z_i) - z_i = f_i, \quad i = 1, \ldots, n.$$ 

with $f_i \in k$. Then $z_1, \ldots, z_n$ are differentially dependent over $k$ if and only if there is a homogeneous linear differential polynomial $L$ over $C$ such that

$$L(z_1, \ldots, z_n) = g \quad g \in k$$

Equivalently, $L(f_1, \ldots, f_n) = \sigma(g) - g$. 

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Corollary. Let $f_1, \ldots, f_n \in \mathbb{C}(x)$, $\sigma(x) = x + 1$, $\delta = \frac{d}{dx}$ and let $z_1, \ldots, z_n$ satisfy

$$\sigma(z_i) - z_i = f_i, \ i = 1, \ldots, n.$$ 

Then $z_1, \ldots, z_n$ are differentially dependent over $\mathcal{F}(x)$ ($\mathcal{F}$ is the field of 1-periodic functions) if and only if there is a homogeneous linear differential polynomial $L$ over $\mathbb{C}$ such that

$$L(z_1, \ldots, z_n) = g \quad g \in \mathbb{C}(x)$$

Equivalently, $L(f_1, \ldots, f_n) = \sigma(g) - g$.

- Similar result for $q$-difference equations. Also for $\sigma y_i = f_i y_i$

- C. Hardouin proved (using difference Galois theory) similar result for $\sigma y_i = f_i y_i$ and gave criterion in terms of divisors of the $f_i$. Simplified by M. van der Put
The Gamma function is hypertranscendental.

- \( z(x) = \frac{\Gamma'(x)}{\Gamma(x)} \) satisfies \( \sigma(z) - z = \frac{1}{x} \).

- If \( z(x) \) satisfies a polynomial differential equation, then
  \[
  \exists L \in \mathbb{C}[\frac{d}{dx}], g(x) \in \mathbb{C}(x) \text{ s.t. } L\left(\frac{1}{x}\right) = g(x + 1) - g(x)
  \]

- \( L\left(\frac{1}{x}\right) \) has a pole \( \Rightarrow \) \( g(x) \) has a pole.

- If \( g(x) \) has a pole then \( g(x + 1) - g(x) \) has at least two poles but \( L\left(\frac{1}{x}\right) \) has exactly one pole.
If $H$ a Zariski-dense proper differential subgroup of $G \subset \text{GL}_n(k)$, a simple algebraic group defined over $C$

$\Rightarrow \exists g \in \text{GL}_n(k)$ such that $gHg^{-1} = G(C)$, $C = \ker(\delta)$.

$\Downarrow$

Proposition. Let $A \in \text{GL}_n(k)$ and assume that the $\sigma$-Galois group of $\sigma(Y) = AY$ is a simple (noncommutative) linear algebraic group $G$ of dimension $t$. Let $R = k\{Z, \frac{1}{\det Z}\}$ be the $\sigma\delta$-PV ring.

The differential trans. deg. of $R$ over $k$ is less than $t$

$\Uparrow$

$\exists B \in \text{gl}_n(k)$ s.t. $\sigma(B) = ABA^{-1} + \delta(A)A^{-1}$

(in which case, $(\delta Z - BZ)Z^{-1} \in \text{gl}_n(k^\sigma)$)
Ex.

\[ k = \mathbb{C}(x), \sigma(x) = x + 1 \]

\[ \sigma^2 y - x \sigma y + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y \]

\[ R = k[Y, \frac{1}{\det(Y)}]/(\det(Y) - 1), \quad \text{Gal}_\sigma = \text{SL}_2(\mathbb{C}) \]

\[ y_1(x), y_2(x) \text{ linearly independent solutions.} \]

\[ y_1(x), y_2(x), y_1(x + 1) \text{ are differentially dependent over } \mathbb{C}(x) \]

\[ \uparrow \]

\[ \exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{gl}_2(\mathbb{C}(x)) \text{ s.t.} \]

\[ \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}' \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}^{-1} + \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}^{-1} \]

This 4\textsuperscript{th} order inhomogeneous equation has no such solutions

\[ \Rightarrow y_1(x), y_2(x), y_1(x + 1) \text{ are differentially independent over } \mathbb{C}(x) \]
Final Comments

- $q$-hypergeometric functions $2\phi_1(a, b; c; x)$ satisfy
  \[
  \phi(q^2x) - \frac{(a - b)x - (1 + c/q)}{abx - c/q} \phi(qx) + \frac{x - 1}{abx - c/q} \phi(x) = 0
  \]

  Classify differential dependence among these. (J. Roques has calculated the (tannakian) Galois groups.)

- Nonlinear equations
- Isomonodromic deformations of difference equations
- Inverse problem
- Relation to model theory of $\sigma \delta$-fields (R. Bustamante)