Differential Groups and Differential Relations

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Theorem: (Hölder, 1887) The Gamma function $\Gamma(x+1) = x\Gamma(x)$ satisfies no polynomial differential equation.

Goal: Prove this using *differential* algebraic groups and generalize

Ex. If $y_1(x)$ and $y_2(x)$ are lin. indep. solutions of

$$y(x+2) - xy(x+1) + y(x) = 0$$

then $y_1(x), y_1(x+1)$ and $y_2(x)$ satisfy no polynomial differential equation.

- To study an object $\mathcal{X}$, study its group of symmetries $\mathcal{G}$

- The size of $\mathcal{G}$, measures the size of $\mathcal{X}$

- The relations defining $\mathcal{G}$ give us the relations on $\mathcal{X}$.

- Galois Theory of Polynomial Equations
- Galois Theory of Difference Equations
- Linear Differential Algebraic Groups
- Differential Galois Theory of Difference Equations
- Differential Relations Among Solutions of Linear Difference Equations
- Final Comments

Galois Theory of Polynomial Equations

$f(y) = 0, f \in k[y]$ of degree n and irreducible

Galois group = the group of transformations of the roots of f that preserve
all algebraic relations among them.

More formally:

Splitting Ring: $K = k[y_1, \ldots, y_n, (\prod_{i<j}(y_i - y_j))^{-1}]/M = k[\alpha_1, \ldots, \alpha_n]$,
M a max ideal containing $(f(y_1), \ldots, f(y_n))$
Note: K is a field and all such are isomorphic.

Galois group = $\mathrm{Gal}(K/k) = \{\sigma : K \to K \mid \sigma$ is a k-autom. $\}$

$$K = k[\alpha_1, \ldots, \alpha_n]$$

$$\alpha = (\alpha_1, \ldots, \alpha_n), \quad V = \{\sigma(\alpha) \mid \sigma \in \mathrm{Gal}(K/k)\} \subset K^n$$

- V is a variety, inv. under $\mathrm{Gal}(\bar{k}/k) \Rightarrow V$ defined over k

  $\mathrm{Gal}(K/k)$ acts trans. and freely on $V \Rightarrow$ V is a $\mathrm{Gal}(K/k)$-torsor

- $K = k[\alpha_1, \ldots, \alpha_n] =$ coordinate ring of V

$$K^{\mathrm{Gal}(K/k)} = k \qquad |\mathrm{Gal}(K/k)| = |V| = \dim_k K$$

The size of $\mathrm{Gal}(K/k)$ measures relations among the roots.

Ex. $|\mathrm{Gal}(K/k)| = \deg(f) \Rightarrow$ all roots are expressed in terms of one.

Galois Theory of Difference Equations

k - field, σ - an automorphism Ex. $\mathbb{C}(x),\ \sigma(x) = x+1,\ \sigma(x) = qx$

Difference Equation: $\sigma(Y) = AY \quad A \in \mathrm{GL}_n(k)$

Splitting Ring: $k[Y, \frac{1}{\det(Y)}]$, $Y = (y_{i,j})$ indeterminates , $\sigma(Y) = AY$,

$$M = \text{ max } \sigma\text{-ideal}$$

$$R = k[Y, \frac{1}{\det(Y)}]/M = k[Z, \frac{1}{\det(Z)}] = \sigma\text{-Picard-Vessiot Ring}$$

- M is radical $\Rightarrow$ R is reduced
- If $C = k^\sigma = \{c \in k \mid \sigma c = c\}$ is alg closed $\Rightarrow$ R is unique and $R^\sigma = C$

Ex.

$$k = \mathbb{C} \quad \sigma(y) = -y$$

$$R = \mathbb{C}[y, \frac{1}{y}]/(y^2 - 1)$$

σ-Galois Group: $\mathrm{Gal}_\sigma(R/k) = \{\phi : R \to R \mid \phi$ is a σ k-automorphism$\}$

Ex.

$$k = \mathbb{C} \quad \sigma(y) = -y \quad \Rightarrow R = \mathbb{C}[y, \tfrac{1}{y}]/(y^2 - 1)$$

$$\mathrm{Gal}_\sigma(R/k) = \mathbb{Z}/2\mathbb{Z}$$

Ex.

$$k = \mathbb{C}(x), \sigma(x) = x + 1$$

$$\sigma^2 y - x\sigma y + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y$$

$$R = k[Y, \tfrac{1}{\det(Y)}]/(\det(Y) - 1), \quad \mathrm{Gal}_\sigma = \mathrm{SL}_2(\mathbb{C})$$

Ex.

$$\sigma(y) - y = f, \; f \in k \Leftrightarrow \sigma \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

$$\phi \in \mathrm{Gal}_\sigma \Rightarrow \phi(y) = y + c_\phi, c_\phi \in C$$

$$\mathrm{Gal}_\sigma = (C, +) \text{ or } \{0\}$$

- $\phi \in \mathrm{Gal}_\sigma,\ \sigma(Z) = AZ \Rightarrow \phi(Z) = Z[\phi],\ \ [\phi] \in \mathrm{GL}_n(C)$

  $\mathrm{Gal}_\sigma \hookrightarrow \mathrm{GL}_n(C)$ and the image is Zariski closed

  $\mathrm{Gal}_\sigma = G(C)$, G a lin. alg. gp. $/C$.

- $R =$ coord. ring of a G-torsor

  $R^{\mathrm{Gal}_\sigma} = k$

  $\dim(G) =$ Krull dim.$_k R$ ($\simeq$ trans. deg. of quotient field)

The size of $\mathrm{Gal}(K/k)$ measures algebraic relations among the solutions .

Ex.

$$\sigma^2 y - x\,\sigma y + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y$$

$$\mathrm{Gal}_\sigma = \mathrm{SL}_2(\mathbb{C})$$

$$3 = \dim \mathrm{SL}_2(\mathbb{C}) = tr.\ deg._k k(y_1, y_2, \sigma(y_1), \sigma(y_2))$$

$\Rightarrow y_1, y_2, \sigma(y_1)$ alg. indep. over k

<u>Ex.</u> $f_1, \ldots, f_n \in k,\ k$ a difference field w. alg. closed const.

$$\sigma(y_1) - y_1 = f_1$$
$$\vdots$$
$$\sigma(y_n) - y_n = f_n$$

Picard-Vessiot ring = $k[y_1, \ldots, y_n]$

<u>Prop.</u> $y_1, \ldots, y_n$ alg. dep. over k

if and only if

$\exists g \in k$ and a const coeff. linear form L s.t. $L(y_1, \ldots, y_n) = g$

(equiv., $c_1 f_1 + \ldots + c_n f_n = \sigma(g) - g$)

<u>Proof.</u> $\mathrm{Gal}_\sigma \subset (C, +)^n$.

$\Rightarrow$) Alg. dependent $\Rightarrow$ $\mathrm{Gal}_\sigma \subsetneq (C, +)^n$

$\Rightarrow$ $\exists\, L$ s.t. $\mathrm{Gal}_\sigma \subset \{(c_1, \ldots, c_n) \mid L(c_1, \ldots, c_n) = 0\}$

$\phi \in \mathrm{Gal}_\sigma,\ \phi(L(y_1, \ldots, y_n)) = L(y_1 + c_1, \ldots, y_n + c_n)$

$= L(y_1, \ldots, y_n) + L(c_1, \ldots, c_n) = L(y_1, \ldots, y_n)$

So, $L(y_1, \ldots, y_n) = g \in k$. $\square$

<u>Ex.</u> $y(x+1) - y(x) = \frac{1}{x} \Rightarrow y(x)$ is not alg. over $\mathbb{C}(x)$.

Linear Differential Algebraic Groups

P. Cassidy-“Differential Algebraic Groups” *Am. J. Math.* 94(1972),891-954
+ 5 more papers, book by Kolchin, papers by Buium, Pillay et al.,
Ovchinnikov

(k, δ) = a differentially closed differential field

Definition: A subgroup $G \subset \mathrm{GL}_n(k) \subset k^{n^2}$ is a linear differential algebraic group if it is Kolchin-closed in $\mathrm{GL}_n(k)$, that is, G is the set of zeros in $\mathrm{GL}_n(k)$ of a collection of differential polynomials in n^2 variables.

Ex. Any linear algebraic group defined over k, that is, a subgroup of $\mathrm{GL}_n(k)$ defined by (algebraic) polynomials, e.g., $\mathrm{GL}_n(k), \mathrm{SL}_n(k)$

Ex. Let $C = \ker \delta$ and let $G(k)$ be a linear algebraic group defined over k. Then $G(C)$ is a linear *differential* algebraic group (just add $\{\delta y_{i,j} = 0\}_{i,j=1}^{n}$ to the defining equations!)

Ex. Differential subgroups of $G_a(k) = (k, +) = \{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in k\}$

The linear differential subgroups are all of the form

$$G_a^L = \{z \in k \mid L(z) = 0\}$$

where L is a linear homogeneous differential polynomial.

For example, if $m = 1$,

$$G_a^\delta = \{z \in k \mid \delta(z) = 0\} = G_a(C)\}$$

Ex. Differential subgroups of $G_a^n(k) = (k^n, +)$

The linear differential subgroups are all of the form

$$G_a^L = \{(z_1, \ldots, z_n) \in k^n \mid L(z_1, \ldots, z_n) = 0\}$$

where L is a linear homogeneous differential polynomial.

Ex. H a Zariski-dense proper differential subgroup of $\mathrm{SL}_n(k)$

$\Rightarrow \exists g \in \mathrm{SL}_n(k)$ such that $gHg^{-1} = \mathrm{SL}_n(C),\ C = \ker(\delta)$.

In general if H a Zariski-dense proper differential subgroup of $G \subset \mathrm{GL}_n(k)$, a simple algebraic group defined over C

$\Rightarrow \exists g \in \mathrm{GL}_n(k)$ such that $gHg^{-1} = G(C),\ C = \ker(\delta)$.

Differential Galois Theory of Difference Equations

k - field, σ - an automorphism $\quad \delta$ - a derivation s.t. $\sigma\delta = \delta\sigma$

<u>Ex.</u> $\mathbb{C}(x): \ \sigma(x) = x + 1, \ \delta = \frac{d}{dx}$

$\sigma(x) = qx, \qquad \delta = x\frac{d}{dx}$

$\mathbb{C}(x,t): \ \sigma(x) = x + 1, \ \delta = \frac{\partial}{\partial t}$

Difference Equation: $\sigma(Y) = AY \ \ A \in \mathrm{GL}_n(k)$

Splitting Ring: $k\{Y, \frac{1}{\det(Y)}\} = k[Y, \delta Y, \delta^2 Y, \ldots, \frac{1}{\det(Y)}]$

$Y = (y_{i,j})$ *differential* indeterminates

$\sigma(Y) = AY, \ \sigma(\delta Y) = A(\delta Y) + (\delta A)Y, \ldots$

$M =$ max $\sigma\delta$-ideal

$$R = k\{Y, \frac{1}{\det(Y)}\}/M = k\{Z, \frac{1}{\det(Z)}\} = \sigma\delta\text{-Picard-Vessiot Ring}$$

$\boldsymbol{k}$ - $\boldsymbol{\sigma\delta}$ field

$$\boldsymbol{\sigma(Y) = AY, A \in \mathrm{GL}_n(k)}$$

$\boldsymbol{R = k\{Z, \frac{1}{\det(Z)}\}}$ - $\boldsymbol{\sigma\delta}$-Picard-Vessiot ring

- $\boldsymbol{R}$ is reduced

- If $\boldsymbol{C = k^\sigma = \{c \in k \mid \sigma c = c\}}$ is *differentially* closed
  $\Rightarrow$ $\boldsymbol{R}$ is unique and $\boldsymbol{R^\sigma = C}$

$\sigma\delta$-Galois Group: $\mathbf{Gal}_{\sigma\delta}(R/k) = \{\phi : R \longrightarrow R \mid \phi$ is a $\sigma\delta$ k-automorphism$\}$

- $\phi \in \mathbf{Gal}_{\sigma\delta}$ $\sigma(Z) = AZ \Rightarrow \phi(Z) = Z[\phi], \;\; [\phi] \in \mathbf{GL}_n(C)$

  $\mathbf{Gal}_{\sigma\delta} \hookrightarrow \mathbf{GL}_n(C)$ and the image is Kolchin closed

  $\mathbf{Gal}_{\sigma\delta} = G(C)$, G a lin. *differential* alg. gp. $/C$.

- $\mathbf{Gal}_{\sigma\delta}$ is Zariski dense in $\mathbf{Gal}_{\sigma}$

- $R =$ coord. ring of a G-torsor

  - $R^{\mathrm{Gal}_{\sigma\delta}} = k$

  - Assume G connected. Then diff. dim.$_C(G) =$ diff. tr. deg$_k F$ where F is the quotient field of R.

Ex.

$$k = \tilde{\mathbb{C}} \ \ \sigma(y) = -y \ \Rightarrow R = k[y, \tfrac{1}{y}]/(y^2 - 1)$$

$$\mathrm{Gal}_{\sigma\delta}(R/k) = \mathbb{Z}/2\mathbb{Z}$$

Ex.

$$\sigma(y) - y = f, \ f \in k, \quad \mathrm{Gal}_{\sigma\delta} \subset \mathbb{G}_a$$

$\Rightarrow \mathrm{Gal}_{\sigma\delta} = \{c \in R^\sigma \mid L(c) = 0\}$ for some $L \in R^\sigma[\delta]$.

Ex.

$$k = \tilde{\mathbb{C}}(x), \sigma(x) = x + 1, \ \delta(x) = 1$$

$$\sigma^2 y - xy + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y$$

Will show: $R = k\{Y, \frac{1}{\det(Y)}\}/\{\det(Y) - 1\}$

$$\mathrm{Gal}_{\delta\sigma} = \mathrm{SL}_2(\tilde{\mathbb{C}})$$

Differential Relations Among Solutions of Linear Difference Equations

Groups Measure Relations

k - $\sigma\delta$ - field, $C = k^\sigma$ differentially closed.

Differential subgroups of $G_a^n(k) = (k^n, +)$ are all of the form

$$G_a^L = \{(z_1, \dots, z_n) \in k^n \mid L(z_1, \dots, z_n) = 0\}$$

where L is a linear homogeneous differential polynomial.

$$\Downarrow$$

Proposition. Let R be a $\sigma\delta$-Picard-Vessiot extension of k containing $z_1, \dots, z_n$ such that

$$\sigma(z_i) - z_i = f_i, \;\; i = 1, \dots, n.$$

with $f_i \in k$. Then $z_1, \dots, z_n$ are differentially dependent over k if and only if there is a homogeneous linear differential polynomial L over C such that

$$L(z_1, \dots, z_n) = g \quad g \in k$$

Equivalently, $L(f_1, \dots, f_n) = \sigma(g) - g$.

Corollary. Let $f_1, \ldots, f_n \in \mathbb{C}(x), \sigma(x) = x + 1, \delta = \frac{d}{dx}$ and let $z_1, \ldots, z_n$ satisfy

$$\sigma(z_i) - z_i = f_i, \;\; i = 1, \ldots, n.$$

Then $z_1, \ldots, z_n$ are differentially dependent over $\mathcal{F}(x)$ ($\mathcal{F}$ is the field of 1-periodic functions) if and only if there is a homogeneous linear differential polynomial L over $\mathbb{C}$ such that

$$L(z_1, \ldots, z_n) = g \quad g \in \mathbb{C}(x)$$

Equivalently, $L(f_1, \ldots, f_n) = \sigma(g) - g$.

- Similar result for q-difference equations. Also for $\sigma y_i = f_i y_i$

- C. Hardouin proved (using difference Galois theory) similar result for $\sigma y_i = f_i y_i$ and gave criterion in terms of divisors of the f_i. Simplified by M. van der Put

The Gamma function is hypertranscendental.

- $z(x) = \Gamma'(x)/\Gamma(x)$ satisfies $\sigma(z) - z = \frac{1}{x}$.

- If $z(x)$ satisfies a polynomial differential equation, then

$$\exists L \in \mathbb{C}[\frac{d}{dx}], g(x) \in \mathbb{C}(x) \text{ s.t. } L(\frac{1}{x}) = g(x+1) - g(x)$$

- $L(\frac{1}{x})$ has a pole $\Rightarrow$ $g(x)$ has a pole.

- If $g(x)$ has a pole then $g(x+1) - g(x)$ has at least two poles but $L(\frac{1}{x})$ has exactly one pole .

If H a Zariski-dense proper differential subgroup of $G \subset \mathrm{GL}_n(k)$, a simple algebraic group defined over C

$$\Rightarrow \exists g \in \mathrm{GL}_n(k) \text{ such that } gHg^{-1} = G(C),\ C = \ker(\delta).$$

$$\Downarrow$$

Proposition. Let $A \in \mathrm{GL}_n(k)$ and assume that the σ-Galois group of $\sigma(Y) = AY$ is a simple (noncommutative) linear algebraic group G of dimension t. Let $R = k\{Z, \frac{1}{\det Z}\}$ be the $\sigma\delta$-PV ring.

The differential trans. deg. of R over k is less than t

$$\Updownarrow$$

$\exists\ B \in \mathrm{gl}_n(k)$ s.t. $\sigma(B) = ABA^{-1} + \delta(A)A^{-1}$
(in which case, $(\delta Z - BZ)Z^{-1} \in \mathrm{gl}_n(k^\sigma)$)

Ex.

$$\boldsymbol{k = \mathbb{C}(x), \sigma(x) = x+1}$$

$$\boldsymbol{\sigma^2 y - x\,\sigma y + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y}$$

$$\boldsymbol{R = k[Y, \tfrac{1}{\det(Y)}]/(\det(Y)-1), \ \ \mathrm{Gal}_\sigma = \mathrm{SL}_2(\mathbb{C})}$$

$y_1(x), y_2(x)$ linearly independent solutions.

$y_1(x), y_2(x), y_1(x+1)$ are differentialy dependent over $\mathbb{C}(x)$

$$\Updownarrow$$

$\exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{gl}_2(\mathbb{C}(x))$ s.t.

$$\boldsymbol{\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}' \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}^{-1} + \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}^{-1}}$$

This 4^{th} order inhomogeneous equation has no such solutions

$\Rightarrow y_1(x), y_2(x), y_1(x+1)$ are differentialy independent over $\mathbb{C}(x)$

Final Comments

- q-hypergeometric functions ${}_2\phi_1(a,b;c;x)$ satisfy

  $$\phi(q^2x)-\frac{(a-b)x-(1+c/q)}{abx-c/q}\phi(qx)+\frac{x-1}{abx-c/q}\phi(x)=0$$

  Classify differential dependence among these. (J. Roques has calculated the (tannakian) Galois groups.)
- Nonlinear equations
- Isomonodromic deformations of difference equations
- Inverse problem
- Relation to model theory of $\sigma\delta$-fields (R. Bustamante)