

# *Model Theory and Differential Algebra*

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## Languages and Structures

In model theory we use first order languages to study mathematical structures.

**Examples** i) Language of rings:  $\mathcal{L}_0 = \{+, -, \cdot, 0, 1\}$

Possible structures:  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ .

ii) Language of differential rings  $\mathcal{L} = \{+, -, \cdot, \delta, 0, 1\}$ .

Possible structures:  $\mathbb{C}(t)$  with  $\delta(t) = 1$ .

## Language and Formulas

We build up *formulas* using the symbols of our language and

- equality:  $=$
- variables:  $x, y, z, \dots$ ;
- Propositional connectives  $\wedge$  (and),  $\vee$  (or),  $\neg$  (not),  $\rightarrow$  (implies);
- Quantifiers:  $\forall, \exists$
- Parenthesis:  $(, )$

## Examples of $\mathcal{L}$ -Formulas

i)  $\forall x \forall y (\delta(xy) = x\delta(y) + y\delta(x))$

ii)  $\delta(x) = 0$

iii)  $\exists y \delta(y) = y\delta(x)$

i) is a declarative sentence expressing a property of the field.

ii) and iii) express properties of the variable  $x$ .

$x$  is *free* in ii) and iii), i.e. not inside a quantifier  $\exists x$  or  $\forall x$ .

## Definable Sets

Let  $K$  be a differential field and let  $\phi$  be a formula with free variables  $x_1, \dots, x_n, y_1, \dots, y_m$ . We write

$$\phi(\bar{x}, \bar{y}).$$

Let  $\bar{a} \in K^m$ . Then  $\phi(\bar{x}, \bar{a})$  defines

$$\{x \in K^n : \phi(\bar{x}, \bar{a}) \text{ holds}\}.$$

For example: Let  $\phi(x, y)$  be

$$\delta(x) = x\delta(y)$$

and  $\alpha \in K$ . Then  $\phi(x, \alpha)$  defines

$$\{x \in K : \delta(x) = x\delta(\alpha)\}$$

## Definable Subsets of $\mathbb{Z}$

Gödel showed quantifiers allow us to build **very** complicated definable sets.

One manifestation is the negative solution to Hilbert's 10th Problem.

**Theorem 1 (MRDP)** *There is  $p(X, Y, \bar{Z}) \in \mathbb{Z}[X, Y, \bar{Z}]$  such that  $p(e, n, \bar{Z}) = 0$  has an integer solution if and only if the  $e^{\text{th}}$  Turing machine halts on input  $n$ .*

## Definable Subsets of $\mathbb{C}$

**Theorem 2 (Tarski)** *In  $\mathbb{C}$  every formula is equivalent to a formula without quantifiers.*

Quantifier free definable sets are Boolean combinations of

$$\{\bar{x} \in \mathbb{C}^n : p(\bar{x}) = 0\}$$

where  $p \in \mathbb{C}[\bar{X}]$ .

Thus Definable = Constructible

= Boolean combinations of Zariski Closed Sets

## Differentially Closed Fields

A differential field  $(K, +, \cdot, \delta)$  is *differentially closed* if for all  $n, m \in \mathbb{N}$  and  $f_1, \dots, f_m \in K\{X_1, \dots, X_n\}$  if the system of equations

$$f_1(X_1, \dots, X_n) = \dots = f_m(X_1, \dots, X_n) = 0$$

has a solution in some differential field  $L \supseteq K$ , then it has one in  $K$ .

I will always assume the characteristic is 0.

## Examples

- There are no natural examples.
- For any differential field  $k$  there is a differentially closed  $K \supseteq k$ .
- By Seidenberg, there are countable differentially closed fields of germs of meromorphic functions.

## Why Do We Study Differentially Closed Fields?

- They provide universal domains for doing differential algebra.
- Many interesting model theoretic phenomena arise in interesting ways.
- Applications to Diophantine Geometry.

## Early Results in the Model Theory of DCF

- Axiomatizability
- Quantifier Elimination
- $\omega$ -Stability
- Differential Closures

## Axioms for Differentially Closed Fields

**Theorem 3 (Blum)** *We can axiomatize the class of differentially closed fields as follows:*

- *Field axioms*
- $\forall x \forall y (\delta(x + y) = \delta(x) + \delta(y) \wedge \delta(xy) = x\delta(y) + y\delta(x))$
- *If  $f, g \in K\{X\}$  and  $\text{ord } f > \text{ord } g$ , there is  $x$  such that  $f(x) = 0 \wedge g(x) \neq 0$ .*

## Quantifier Elimination

**Theorem 4 (Robinson)** *For every formula  $\phi$  there is a quantifier free formula  $\psi$  equivalent in every differentially closed field.*

Quantifier free definable sets are Boolean combinations of Kolchin-closed sets

$$\{\bar{x} \in K^n : f(\bar{x}) = 0\}$$

where  $f \in K\{X_1, \dots, X_n\}$ .

We might call these  $\delta$ -constructible.

## Types and $\omega$ -stability

If  $k \subseteq K$  and  $\bar{a} \in K^n$ ,  $\text{tp}(\bar{a}/k)$ , the *type* of  $\bar{a}$  over  $k$  is the collection of all subsets  $X \subseteq K^n$  defined with parameters from  $k$  such that  $\bar{a} \in X$ .

By quantifier elimination,  $\text{tp}(\bar{a}/k)$  is determined by the prime differential ideal

$$\{f \in k\{X_1, \dots, X_n\} : f(\bar{a}) = 0\}.$$

By the  $\delta$ -Basis Theorem, there are at most  $|k|$  such ideals.

**Corollary 5 (Blum)** *Differentially closed fields are  $\omega$ -stable.*

## Differential Closures

We say that a differentially closed  $K \supseteq k$  is a *differential closure* of  $k$ , if for any differentially closed  $L \supseteq k$ , there is a differential field embedding  $f : K \rightarrow L$  fixing  $k$ .

**Theorem 6** *i) Every differential field has a differential closure.*

*ii) The differential closure is unique up to isomorphism.*

These follow immediately from general results of Morley and Shelah on  $\omega$ -stable theories.

**Note:** The differential closure need not be minimal. Possibly,  $K$  is isomorphic to a proper differential subfield of itself.

## The Modern Theory: Strongly Minimal Sets

In a general structure  $M$  we say that a definable  $X \subseteq M^n$  is *strongly minimal* if  $X$  is infinite, but there is no definable  $Y \subseteq X$  such that  $Y$  and  $X \setminus Y$  are infinite.

Examples:

- A set with no structure.
- A set with an equivalence relation  $E$  where class has two elements.
- An infinite set  $X$  and a bijection  $f : X \rightarrow X$  with no cycles.

## Algebraic Strongly Minimal Sets

- A vector space over  $\mathbb{Q}$ .
- An algebraically closed field
- Any irreducible curve in an algebraically closed field.

In Differentially closed fields (the easiest to see)

- $\{x \in K : f(x) = 0\}$ ,  $f \in K\{X\}$  irreducible of order 1
- $C = \{x : \delta(x) = 0\}$ .

## Combinatorial Geometry

On any strongly minimal set there is a closure operator

$$\text{cl} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$$

defined by  $b \in \text{cl}(A)$  if there is a formula  $\phi(x, \bar{y})$  and  $\bar{a} \in A$  such that

$$\{x : \phi(x, \bar{a})\} \text{ is finite}$$

and  $\phi(b, \bar{a})$ .

- A set with no structure.

$$b \in \text{cl}(A) \Leftrightarrow b \in A$$

- A set with an equivalence relation  $E$  where class has two elements.

$$b \in \text{cl}(A) \Leftrightarrow \exists a \in A \ aEb$$

- An infinite set  $X$  and a bijection  $f : X \rightarrow X$  with no cycles.

$$b \in \text{cl}(A) \Leftrightarrow b \text{ is reachable from } A.$$

- A vector space over  $\mathbb{Q}$ .

$$b \in \text{cl}(A) \Leftrightarrow b \text{ is in the } \mathbb{Q} - \text{span of } A.$$

- An algebraically closed field

$$\text{cl} = \text{algebraic closure}$$

- Differentially closed fields

$$\text{cl}(B) = \text{algebraic closure of differential field generated by } B$$

## Trichotomy

Let  $X$  be strongly minimal

- $X$  is *trivial* if  $\text{cl}(A) = \bigcup_{a \in A} \text{cl}(a)$

Examples: sets with no structure, equivalence relation, cycle-free bijection

- $X$  is *modular* if  $c \in \text{cl}(A \cup \{b\}) \Leftrightarrow \exists a \in \text{cl}(A) \ c \in \text{cl}(\{a, b\})$

In a  $\mathbb{Q}$ -vector space if

$$c = m_1 a_1 + \dots + m_n a_n + mb,$$

let  $a = m_1 a_1 + \dots + m_n a_n$ .

Note: Fields are non-modular.

## Zilber's Paradise

Zilber conjectured that:

i) non-trivial modular strongly minimal sets always arise from a group.

ii) non-modular strongly minimal sets always arise from a field.

**Theorem 7 (Hrushovski)** *i) is true*

*ii) is false*

While ii) is not true in complete generality in many settings it is true.

## Strongly Minimal Sets in DCF

In differentially closed fields all three types of strongly minimal sets arise.

The constant field  $C = \{x : \delta(x) = 0\}$  is an algebraically closed field and if  $X \subseteq K^n$  is definable,  $X \cap C$  is constructible in  $C$ .

**Theorem 8 (Hrushovski-Zilber)** *Any non-modular strongly minimal set in a differentially closed field is essentially  $C$ .*

A simpler proof was given by Pillay and Ziegler.

## Orthogonality

We need a notion of strongly minimal sets being “the same” .

Let  $X$  and  $Y$  be strongly minimal sets in a differentially closed field  $K$ . We say that  $X$  and  $Y$  are *orthogonal* and write  $X \perp Y$  if there is a differentially closed  $L \supset K$  with  $a \in X(L)$  such that  $Y(L) = Y(K)$ .

$X$  and  $Y$  are “the same” if  $X \not\perp Y$ .

If  $K$  is an algebraically closed field and  $X \subseteq K^n$  is strongly minimal then  $X \not\perp K$ .

## Modular Strongly Minimal Sets

Let  $A$  be an Abelian variety defined over  $K$  of dimension  $d$ .

**Theorem 9 (Buium–Manin)** *There is a differential algebraic homomorphism  $\mu : A \rightarrow K^n$  such that the kernel of  $\mu$  is  $A^\sharp$  the Kolchin-closure of the torsion points.*

For example, if  $E$  is an elliptic curve

$$Y^2 = X^3 + aX + b$$

where  $a, b \in C$ , then

$$\mu(x, y) = \frac{\delta(x)}{y}.$$

In general, if  $A$  is defined over  $C$ , then  $\mu$  is the logarithmic-derivative and  $A^\sharp = A(C)$ .

$A$  is *simple* if there is no proper infinite sub-Abelian variety.

**Theorem 10 (Hrushovski–Sokolović)** *If  $A$  is a simple Abelian variety not isomorphic to a variety defined over  $C$ , then  $A^\sharp$  is strongly minimal.*

*If  $A$  and  $B$  are two such simple Abelian varieties, then  $A^\sharp \not\cong B^\sharp$  if and only if  $A$  and  $B$  are isogenous.*

*Moreover, every non-trivial modular strongly minimal set  $X$  in a differentially closed field, there is an Abelian variety  $A$  as above with  $X \not\cong A^\sharp$ .*

## A Simple Diophantine Application

**Corollary 11** *If  $A$  is a simple Abelian variety not defined over  $C$  and  $X \subset A$  is a proper subvariety, then  $X$  contains finitely many torsion points.*

**Proof:** Since  $A^\#$  is strongly minimal  $X \cap A^\#$  is either finite or cofinite in  $A^\#$ . If  $X$  contains infinitely many torsion points it will contain all but finitely many torsion points and be Zariski dense in  $A$ , a contradiction.

## Trivial Sets in Differentially Closed Fields

There are many ad hoc examples of trivial strongly minimal sets.

The proofs by Kolchin, Rosenlicht and Shelah of non-minimality of differential closures show that

$$\delta(X) = X^3 - X$$

and

$$\delta(X) = \frac{X}{X+1}$$

are basically infinite sets with no structure.

**Theorem 12 (Rosenlicht)** *Let  $f(X) \in C(X)$ . The set  $\delta(x) = f(x)$  is non-orthogonal to the constants if and only if*

$$\frac{1}{f(X)} = c \frac{\frac{\partial g(X)}{\partial X}}{g(X)} \text{ or } \frac{\partial g(X)}{\partial X}$$

*for some  $c \in C$  and  $g(X) \in C(X)$ . Otherwise,  $\delta(x) = f(X)$  is a trivial strongly minimal set with no structure.*

## Families of Examples

Let  $C$  be a curve of genus  $g \geq 2$  defined over the constants.

Let  $\omega$  be a regular differential form on  $C$ .

We view  $\omega : T(C) \rightarrow K$ .

The derivation gives us  $\nabla : C \rightarrow T(C)$  a section of the tangent bundle.

Let  $(C, \omega)^\# = \{x \in C : \omega(\nabla(C)) = 1\}$ .

**Theorem 13 (Hrushovski–Itai)** *Almost always  $(C, \omega)^\#$  is a trivial strongly minimal set—indeed an infinite set with no structure.*

Rosen has some generalizations when  $C$  is not defined over the constants. In this case we need to use prolongations rather than tangent bundles.

## $\aleph_0$ -categoricity

**Basic Problem** Classify the trivial strongly minimal sets in differentially closed fields

If  $X$  is strongly minimal, then the cl-dimension of  $X$  is the cardinality of the smallest set  $B \subseteq X$  such that  $\text{cl}(B) = X$ .

For example, infinite sets with no structure or equivalence relations with infinitely many classes all of size 2 have cl-dimension 2.

The cycle-free bijections might be finite dimensional.

**Conjecture:** Every trivial strongly minimal set in a differentially closed field is infinite dimensional.

**Theorem 14 (Hrushovski)** *This is true for  $f(x) = 0$  where  $f$  has order 1.*

A key step in Hrushovski's proof is the following result based on a Theorem of Jounolou on differential forms.

**Theorem 15 (Hrushovski)** *Suppose  $V$  is an  $n$ -dimensional Kolchin closed set defined over  $C$  (i.e. the transcendence degree of the differential coordinate ring is  $n$ ). Either*

*i) there are only finitely many Kolchin closed subvarieties of dimension  $n - 1$*

*ii) there is a non-trivial  $\delta$ -rational  $f : V \rightarrow C$ .*

## How Generic are Strongly Minimal Sets

Take  $V \subseteq K^{n+m}$  be Kolichin closed set. For  $a \in K^m$  let

$$V_a = \{x \in K^n : (x, a) \in V\}.$$

Let  $X = \{a : V_a \neq \emptyset\}$ . Suppose  $V_a$  is finite dimensional for all  $a \in X$ .

**Conjecture** For most  $a \in X$ ,  $V_a$  is a trivial strongly minimal set.

For example, let  $V$  be  $p(x, \bar{y}) = 0$  where  $p$  is not linear in  $X$ . What happens?

## Other Topics in the Model Theory of Differential Fields

### Differential Galois Theory

- Poizat's approach to Kolchin's Galois theory of strongly normal extensions
- Pillay's generalized differential Galois theory with finite dimensional differential algebraic Galois groups

## Model Theoretic Dimensions

There are natural algebraic notions of dimension.

- transcendence degree
- differential transcendence degree
- differential Krull dimension (an ordinal dimension)

In any  $\omega$ -stable theory there are model theoretic ordinal dimensions

- Morley rank
- Lascar rank

In algebraically closed fields these notions are all the same.  
In differentially closed fields they are all different.

Model theoretic notions can be very useful in some contexts.