The Complete Picard–Vessiot Closure of the Constants

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F a differential field = field + derivation $D = D_F$

$D(a+b) = Da + Db$

$D(ab) = aDb + (Da)b$

$C = \text{Ker}(D) \subseteq F$ field of constants, assume algebraically closed, characteristic 0.

Examples

C, $D_C = 0$

$C(x)$, $D = \frac{d}{dx}$, $D(x) = 1$

$C(z)$, $Dz = z$ ($z = \exp(x)$)

$C(z_\alpha)$, $Dz_\alpha = \alpha z_\alpha$ ($z_\alpha = \exp(\alpha x)$)

Differential Equations over F

Differential Operator $L(Y) = Y^{(n)} + a_{n-a}Y^{(n-1)} + \cdots + a_0 y^{(n)}$, $a_i \in F$

Equation $L(Y) = 0$

Solution of $L(Y) = 0$:

$E \supset F$ differential field extension, $y \in E$, $L(y) = 0$.

Full set of solutions: $y_1, \ldots, y_n \in E$, linearly independent over constants

$E \supset F$ is a differential Galois (Picard–Vessiot) extension for L if

- E contains a full set of solutions for $L = 0$.
- E is generated over F as a differential field by these solutions.
- Constants of E= constants of F.

Turns out E is finitely generated over F as a field.

Always exists, unique up to isomorphism.

Galois Theory $E \supset F$

$V = \{y \in E \mid L(y) = 0\}$ n–dimensional vector space over C.

$G(E/F) =$ differential automorphisms of E over F.

$\sigma \in G$ implies $\sigma(V) \subseteq V$, σ a C linear transformation of V

$G \to GL(V)$ injection, image an algebraic group

$\exists$ Galois Correspondence

{Intermediate Differential fields} $\leftrightarrow$ {closed subgroups}

$K \to G(E/K)$, $E^H \leftarrow H$

Example

$$F = C(x),\ L(Y) = Y'' + x^{-1}Y'$$

$$(\log x)' = x^{-1},\ (\log x)'' = -x^{-2},\ -x^{-2} + x^{-1}x^{-1} = 0.$$

$$E = F\langle 1, \log x\rangle = F(\log x) = C(x, \log x)$$

$$V = C1 + C\log x$$

$$\sigma \in G\ \ (\sigma(\log x))' = \sigma((\log x)') = \sigma(x^{-1}) = x^{-1}$$

$$(\log x - \sigma(\log x))' = 0,\ \sigma(\log x) = c(\sigma) + \log x,\ c(\sigma) \in C$$

$$c(\sigma\tau) = c(\sigma) + c(\tau)$$

$$G \to GL(V) = GL_2(C)$$

$$\sigma \mapsto \begin{bmatrix} 1 & c(\sigma) \\ 0 & 1 \end{bmatrix}$$

$F \subseteq \overline{F}$ Differential Galois Closure

Example $F = C$

$\overline{F} \supseteq C(x, \{z_\alpha \mid \alpha \in C\})$, $Dx = 1$, $Dz_\alpha = \alpha z_\alpha$

$L(Y) = \prod (D - \alpha_i)^{n_i}$

For $(D - \alpha_i)^{n_i}$, $\alpha_i \neq 0$

$\{t^k (k!)^{-1} z_{\alpha_i} \mid 0 \leq k \leq n_i - 1\}$ full set of solutions

(So $\overline{C} = C(x, \{z_\alpha \mid \alpha \in C\})$

$G(\overline{F}/F)$:

$\sigma(x) = x + c(\sigma)$, $c(\sigma) \in C$

$\sigma(z_\alpha) = d(\sigma) z_\alpha$, $d(\sigma) \in C - \{0\}$

Note that $\log x \notin \overline{C}$, so $\overline{C}$ has differential Galois extensions

$C \subset \overline{C} \subset \overline{\overline{C}} \subset \ldots$

Notation $F_0 = F$, $F_1 = \overline{F}$, ..., $F_{i+1} = \overline{F_i}$

$\cup_i F_i = F_\infty$

Theorem

- F_∞ has no proper differential Galois extensions.
- F_∞ contains F.
- F_∞ has the same constants as F.

and F_∞ is minimal over F with respect to these properties. This characterizes F_∞ up to F isomorphism.

Automorphisms lift:

$G(F_\infty/F) \twoheadrightarrow G(F_i/F)$

$G(F_\infty/F) = \varprojlim G(F_i/F)$

$G(F_{i+1}/F_i) \hookrightarrow G(F_{i+1}/F) \twoheadrightarrow G(F_i/F)$

$G(F_{i+1}/F_i)$ (pro) algebraic

Example

$C \subset C_\infty$ $F = C(x) \subset C_\infty$ "normal"

$E_\alpha = F(y, w_\alpha) = C(x, y, w_\alpha)$

$y = \log x,\ w_\alpha = \log((\log x) + \alpha)$

$\sigma \in G(C_\infty/F)$

$$y^\sigma = y + b(\sigma), b(\sigma\tau) = b(\sigma) + b(\tau)$$

$$w_\alpha^\sigma = w_{\alpha+b(\sigma)} + c(\sigma, \alpha)$$
$$c(\sigma\tau, \alpha) = c(\sigma, \alpha + b(\tau)) + c(\tau, \alpha)$$

$E = C(x, y, \{w_\alpha \mid \alpha \in C\})$ E "normal"

$G(C_\infty/F) \twoheadrightarrow G(E/F)$

$E = C(x, y, \{z_\alpha \mid \alpha \in C\})$

$G(E/F) \hookrightarrow \mathrm{Map}(C, \mathbb{G}_a) \rtimes \mathbb{G}_a$

$\sigma \mapsto (c(\sigma, \cdot), b(\sigma))$

$\mathbb{G}_a$ acts on $\mathrm{Map}(C, \mathbb{G}_a)$

$(a \cdot f)(c) = f(c + a)$

$\mathrm{Map}(C, \mathbb{G}_a) \rtimes \mathbb{G}_a = \mathbb{G}_a \int_r \mathbb{G}_a$