The Complete Picard–Vessiot Closure of the Constants

Andy Magid
University of Oklahoma
$F$ a differential field = field + derivation $D = D_F$

$D(a + b) = Da + Db$

$D(ab) = aDb + (Da)b$

$C = \text{Ker}(D) \subseteq F$ field of constants, assume algebraically closed, characteristic 0.

**Examples**

$C, \; D_C = 0$

$C(x), \; D = \frac{d}{dx}, \; D(x) = 1$

$C(z), \; Dz = z \; (z = \exp(x))$

$C(z_\alpha), \; Dz_\alpha = \alpha z_\alpha \; (z_\alpha = \exp(\alpha x))$

**Differential Equations over** $F$

Differential Operator $L(Y) = Y^{(n)} + a_{n-a}Y^{(n-1)} + \cdots + a_0y^{(n)}, \; a_i \in F$

Equation $L(Y) = 0$
Solution of \( L(Y) = 0 \):

\( E \supset F \) differential field extension, \( y \in E, \ L(y) = 0 \).

Full set of solutions: \( y_1, \ldots, y_n \in E \), linearly independent over constants

\( E \supset F \) is a differential Galois (Picard–Vessiot) extension for \( L \) if

- \( E \) contains a full set of solutions for \( L = 0 \).
- \( E \) is generated over \( F \) as a differential field by these solutions.
- Constants of \( E \) are constants of \( F \).

Turns out \( E \) is finitely generated over \( F \) as a field.

Always exists, unique up to isomorphism.
**Galois Theory** \( E \supset F \)

\[ V = \{ y \in E \mid L(y) = 0 \} \] \( n \)-dimensional vector space over \( C \).

\( G(E/F) = \) differential automorphisms of \( E \) over \( F \).

\( \sigma \in G \) implies \( \sigma(V) \subseteq V \), \( \sigma \) a \( C \) linear transformation of \( V \)

\( G \to GL(V) \) injection, image an algebraic group

\( \exists \) Galois Correspondence

\{Intermediate Differential fields\} \( \leftrightarrow \) \{closed subgroups\}

\( K \to G(E/K) \), \( E^H \leftarrow H \)
Example

\[ F = C(x), \quad L(Y) = Y'' + x^{-1}Y' \]

\[(\log x)' = x^{-1}, \quad (\log x)'' = -x^{-2}, \quad -x^{-2} + x^{-1}x^{-1} = 0.\]

\[ E = F(1, \log x) = F(\log x) = C(x, \log x) \]

\[ V = C1 + C \log x \]

\[ \sigma \in G \quad (\sigma(\log x))' = \sigma((\log x)') = \sigma(x^{-1}) = x^{-1} \]

\[(\log x - \sigma(\log x))' = 0, \quad \sigma(\log x) = c(\sigma) + \log x, \quad c(\sigma) \in C \]

\[ c(\sigma\tau) = c(\sigma) + c(\tau) \]

\[ G \rightarrow GL(V) = GL_2(C) \]

\[ \sigma \mapsto \begin{bmatrix} 1 & c(\sigma) \\ 0 & 1 \end{bmatrix} \]
$F \subseteq \overline{F}$ Differential Galois Closure

**Example** $F = C$

$\overline{F} \supseteq C(x, \{z_\alpha \mid \alpha \in C\})$, $Dx = 1$, $Dz_\alpha = \alpha z_\alpha$

$L(Y) = \prod (D - \alpha_i)^{n_i}$

For $(D - \alpha_i)^{n_i}, \alpha_i \neq 0$

$\{t^k(k!)^{-1}z_{\alpha_i} \mid 0 \leq k \leq n_i - 1\}$ full set of solutions

(So $\overline{C} = C(x, \{z_\alpha \mid \alpha \in C\})$)

$G(\overline{F}/F)$:

$\sigma(x) = x + c(\sigma), \ c(\sigma) \in C$

$\sigma(z_\alpha) = d(\sigma)z_\alpha, \ d(\sigma) \in C - \{0\}$

Note that $\log x \notin \overline{C}$, so $\overline{C}$ has differential Galois extensions

$C \subset \overline{C} \subset \overline{C} \subset \ldots$
Notation $F_0 = F$, $F_1 = \overline{F}$, \ldots, $F_{i+1} = \overline{F_i}$

$\bigcup_i F_i = F_\infty$

**Theorem**

- $F_\infty$ has no proper differential Galois extensions.
- $F_\infty$ contains $F$.
- $F_\infty$ has the same constants as $F$.

and $F_\infty$ is minimal over $F$ with respect to these properties. This characterizes $F_\infty$ up to $F$ isomorphism.

**Automorphisms lift:**

$G(F_\infty/F) \rightarrow G(F_i/F)$

$G(F_\infty/F) = \lim \leftarrow G(F_i/F)$

$G(F_{i+1}/F_i) \hookrightarrow G(F_{i+1}/F) \rightarrow G(F_i/F)$

$G(F_{i+1}/F_i)$ (pro) algebraic
Example

$C \subset C_{\infty} \ F = C(x) \subset C_{\infty} \ \text{“normal”}$

$E_{\alpha} = F(y, w_{\alpha}) = C(x, y, w_{\alpha})$

$y = \log x, \ w_{\alpha} = \log((\log x) + \alpha)$

$\sigma \in G(C_{\infty}/F')$

$$y^\sigma = y + b(\sigma), b(\sigma \tau) = b(\sigma) + b(\tau)$$

$$w_{\alpha}^\sigma = w_{\alpha + b(\sigma)} + c(\sigma, \alpha)$$

$$c(\sigma \tau, \alpha) = c(\sigma, \alpha + b(\tau)) + c(\tau, \alpha)$$

$E = C(x, y, \{w_{\alpha} \mid \alpha \in C\}) \ E \ \text{“normal”}$

$G(C_{\infty}/F') \twoheadrightarrow G(E/F')$
\[ E = C(x, y, \{ z_\alpha \mid \alpha \in C \}) \]

\[ G(E/F) \hookrightarrow \text{Map}(C, \mathbb{G}_a) \rtimes \mathbb{G}_a \]

\[ \sigma \mapsto (c(\sigma, \cdot), b(\sigma)) \]

\( \mathbb{G}_a \) acts on \( \text{Map}(C, \mathbb{G}_a) \)

\[ (a \cdot f)(c) = f(c + a) \]

\( \text{Map}(C, \mathbb{G}_a) \rtimes \mathbb{G}_a = \mathbb{G}_a \int_r \mathbb{G}_a \)